18.325 Applied Extremal Combinatorics	**Fall 1996**

Lecture 3 : September 16, 1996

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1 Introduction

Last lecture, we discussed the relationship between a graph's second eigenvalue and the density of the sparsest cut of the graph. In this lecture, we'll derive a new bound on λ_2 for planar graphs.

First we introduce the notion of a collection of *kissing disks* (sometimes called a circle packing): this is a collection of disks on the plane whose interiors are all disjoint. The *nerve* of a circle packing is the graph which has a node for every disk; there is an edge between two nodes iff the the corresponding disks intersect.

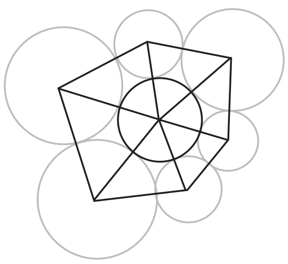
The first topic of today's lecture is a proof of the surprising Koebe embedding theorem.

Theorem 1 *For every planar graph G, there exists a circle packing whose nerve is G.*

The proof will not be constructive.

2 Fully Triangulated Graphs

First we note that it suffices to prove the theorem for fully triangulated planar graphs (a planar graph is *fully triangulated* if all of its faces are triangles.) Suppose the theorem is true for fully triangulated graphs.



Given a planar graph with a face that is not a triangle, we can add a new vertex at the center of the face and edges to all of the nodes on the face. (Note that we're doing this triangulation to the graph, not to any particular embedding of it, so that the edges we're adding don't have to be straight lines.) By hypothesis, a circle embedding exists. Now remove the center disk, and the added vertex and edges to obtain an embedding for the original graph.

How many faces does a triangulated graph have? From the Euler characteristic we have:

$$|F| - |E| + |V| = 2$$

Since the graph is triangulated each face has three edges, and each edge is counted in two faces, so $3|F| = 2|E|$, which gives $|F| = 2|V| - 4$. (Note that if the graph is not fully triangulated, this becomes $|F| \leq 2|V| - 4$.

We will assume that the graph G is "circumtriangulated" so that the outside face is an equilateral triangle.

3 The map σ

Suppose we are given the radius r_i for the disk around node i. The radii of the disks corresponding to the corners of a triangle determine the angles of that triangle. Let σ_i be the sum of the angles of the triangles around vertex i, given the radii $r_1, \ldots, r_n$.

For an interior node i, a graph embedding will have $\sigma_i = 2\pi$. For a boundary node, we want $\sigma_i = \frac{2\pi}{3}$. If this seems strange, think of a boundary vertex as having a contribution of $\frac{\pi}{3}$ from the internal angles, and an additional contribution of $\frac{\pi}{3}$ from the outside face.

In other words, we want to find a vector $\bar{r}$ so that:

$$\sigma(\bar{r}) = (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, 2\pi, 2\pi, \cdots, 2\pi) = \bar{w}.$$

A geometric argument which we'll skip shows that specifying the radii of all of the disks and all of the angle-sums completly specifies an arrangement of triangles on the plane.

We will argue that such an $\bar{r}$ must exist. First, define the polytopes S and P:

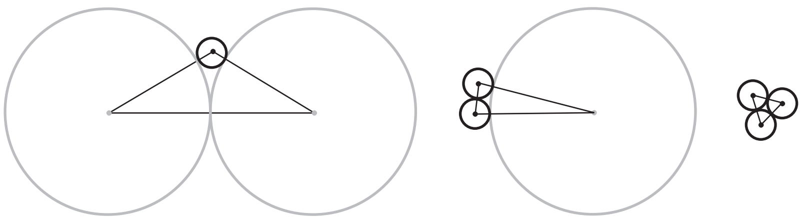
$$\begin{aligned} S &= \{(r_1, \ldots, r_n) | \sum r_i = 1; r_i > 0, \forall i\} \\ P &= \left\{ (\sigma_1, \ldots \sigma_n) \mid \begin{array}{l} \sum \sigma_i = (2n-4)\pi \\ \forall I \subset \{1, \ldots, n\}, \sum_{i \in I} \sigma_i < |F(I)|\pi \end{array} \right\} \end{aligned}$$

Where $F(I)$ is the set of faces meeting an index set I.

We want to prove that σ is surjective. This would be easy if we could extend σ to a continuous function on the boundary of S; but, this is not possible. Instead, we will show that σ is surjective by showing that it takes the boundary of S to the boundary of P and that it is injective.

Claim 1 $\sigma : Bd\,S \to Bd\,P$.

We want to show that as $\bar{r} \to \mathrm{Bd}\,S$, $\sigma(\bar{r}) \to \mathrm{Bd}\,P$. Let I be the set of nodes whose radii is tending towards 0. We will show that $\Sigma_{i \in I} \sigma_i \to |F(I)|\pi$. Suppose $F \in F(I)$. There are three cases, depending on how many of the vertices of F are in I:



(1) (2) (3)

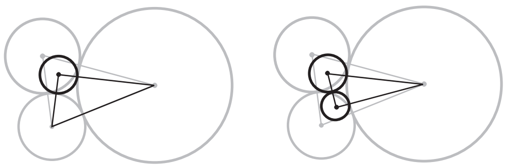
Case 1 The angle at $i \to \pi$. **Case 2** Note that the radii of the two dark circles may approach zero at different rates, so that we don't know anything about the individual angles. Nevertheless, their sum must approach π. **Case 3** All three vertices are in I, the sum remains π.

Claim 2 σ *is injective.*

Suppose we have $r \neq r'$, define $I = \{i \mid r_i < r_i'\}$. Then we claim that

$$\sum_{i \in I} \sigma_i(r) > \sum_{i \in I} \sigma_i(r')$$

Imagine taking r' and altering it to look like r. Again, we'll look at a single face; there are three cases depending on how many of vertices of this face are in I:



(1) (2)

Case 1 The angle at i increases as r_i shrinks. **Case 2** If the radii of disks i and j shrink, then the sum of the angles must i and j increase. **Case 3** If all of the radii shrink, the sum of the angles remains π. However, since $\sum_i r_i = 1$, not all radii can shrink.

The fact that σ is surjective now follows from a theorem of topology. (If $\sigma : S \to T$ is a continuous, injective, mapping from d-dimensions to d-dimensions, and if $\{x_i\}$ is a sequence of points approaching the boundary of S, then $\{\sigma(x_i)\}$ approaches the boundary of T; then σ is surjective.) We're now almost done: we just have to show that $\bar{w} \in P$, and then from the surjectivity of σ, we'll know that its preimage exists, and yields the desired configuration.

Claim 3 $\bar{w} = (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, 2\pi, 2\pi, \cdots, 2\pi) \in P$

We need to check two conditions: (1) $\sum w_i = (n-3)2\pi + 3\frac{2\pi}{3} = (2n-4)\pi$. (2) $\sum_{i \in I} w_i < |F(I)|\pi$. First note that $|F(I)| > 2|I|$: take any subset I; delete all the nodes of I and the incident edges. The resulting graph has $n - |I|$ nodes and has at most $2(n - |I|) - 4$ faces, none of which can be in $F(I)$. Therefore $|F(I)| > 2|I|$. But now, we know $\sum w_i \leq 2\pi|I| < |F(I)|\pi$.

This concludes the proof of the Koebe embedding theorem.

4 Estimating λ_2

We can now use the Koebe embedding theorem to obtain some approximations of the value of λ_2 for planar graphs. The rest of today's lecture is a proof of the following result:

Theorem 2 *If G is a planar graph, then $\lambda_2(G) < 8d/n$ where d is maximum degree of G.*

From the Raleigh quotient characterization of λ_2 we have:

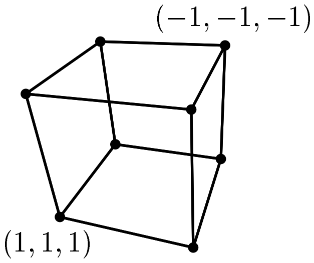
$$\lambda_2 = \min_{\sum x_i = 0} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_i x_i^2}$$

As we showed earlier, this relates to a way of embedding the graph on the line; today we'll look at embedding the graph in space: replace x_i with $\bar{v}_i$.

Prop 1

$$\lambda_2 = \min_{\sum \bar{v}_i = 0} \frac{\sum_{(i,j)\in E} d(\bar{v}_i, \bar{v}_j)^2}{\sum_i \langle \bar{v}_i, \bar{v}_i \rangle}.$$

Example



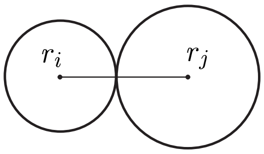
This is an embedding of the cube which yields λ_2. Note that $\sum \bar{v}_i = 0$.

To prove the proposition note that if you split the sums into each dimension, you get the Raleigh quotient of the matrix:

$$\begin{pmatrix} L & & & & \\ & L & & & \\ & & L & & \\ & & & \ddots & \\ & & & & L \end{pmatrix}$$

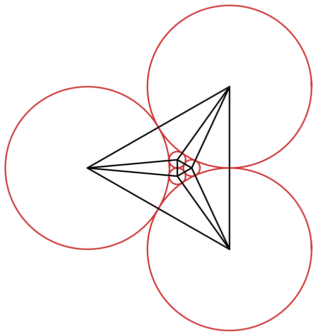
which has the same eigenvalues as the matrix L with multiplicity d.

We can now use the embedding theorem as a way to estimate the Raleigh quotient. Given two touching disks at nodes i and j:



$$d(v_i, v_j)^2 = (r_i + r_j)^2 \leq 2r_i^2 + 2r_j^2$$

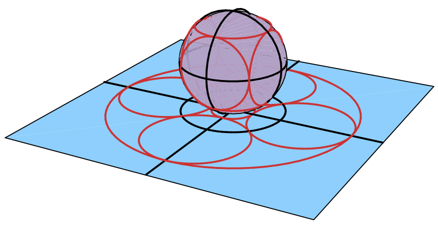
It turns out that if we do this approximation in the plane, there are some bad examples for which the radii of the to circles decrease in geometric progression. One such bad example is the graph consisting of n triangles, nested around the origin. With $n = 2$, the graph and its circle embedding is:



For this graph, our estimate approaches a constant, while the true value of λ_2 is $\Theta(\frac{1}{n^2})$.

We'll deal with this problem by stereographically projecting the circle packing to the sphere. (On the sphere, the denominator in the Raleigh quotient is $\sum_i \langle v_i, v_i \rangle = n$.) Stereographic projection is circle-preserving, so a circle packing on the plane corresponds to one on the sphere. (However, distances are *not* preserved, so that the center of a circle on the sphere does not map to the center of the projected circle.)

A circle packing on the sphere and its stereographic projection to the plane:



Note that our estimate $d(v_i, v_j)^2 \leq 2r_i^2 + 2r_j^2$ still holds for the disks on the sphere if one defines the radius of disk i to be the Euclidean distance from the center of the disk on the sphere to the boundary of the disk.

In the sum in the numerator, each r_i is counted once for each edge leaving node i, which gives us:

$$\sum_{(i,j)\in E} d(v_i, v_j)^2 \leq d \cdot 2 \cdot \sum_i r_i^2$$

The area of disk i is πr_i^2, and the entire packing fits on the sphere, whose area is 4π, so we have:

$$\pi \sum_i r_i^2 \leq 4\pi$$

.

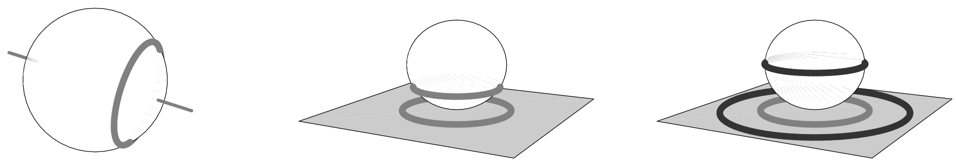
One problem remains: the minimum in the Raleigh quotient is over configurations with $\sum_i v_i = 0$. In our context, this means that the center of gravity of the circle centers must be the origin. In general this will not be the case. However we'll show that it is possible to move the circle packing on the sphere in order to make this true.

Once we've established this, we will have shown that:

$$\lambda_2 = \min_{\sum v_i = 0} \frac{\sum_{(i,j)\in E} d(v_i, v_j)^2}{\sum \langle v_i, v_i \rangle} \leq \frac{8d}{n}.$$

5 Moving the Packing

Consider the following map, which we'll call f. Take a point p in the interior of the sphere. Joining p to the origin determines a line l, and also a circle C on the sphere, normal to l, whose center is p (a). Now use the axis l to stereographically project the sphere onto the plane. The projection of C is a circle on the plane (b); now dilate the plane so that a reverse projection maps the image of C back to the equator (c). Define $f(p)$ to be the center of gravity of the circle centers under this transformation.



(a) (b) (c)

We have defined f as a map from the interior of the sphere to itself. Ideally, we would show that f takes the boundary to the boundary, and in fact takes the point p to $-p$. We could then derive from the Brouwer fixed-point theorem (see below) that its image must include the origin.

This argument is not quite true because the map f doesn't necessarily send points on the boundary to the boundary. But, one can make the argument rigorous by showing that f maps points on the boundary to points near the boundary.

6 The Brouwer fixed-point theorem

Theorem 3 *If $g : B^n \to B^n$ is continuous, there exists a point x such that $g(x) = x$.*

Now suppose we have a continuous antipodal map like f, taking x to $-x$, whose image does not include the origin. Then consider the composition of f with the map that projects the interior of the ball less the origin to the sphere (also known as the 'smush' map). This composition yields a continuous function from B^n to itself with no fixed points, contradicting the theorem. Thus our assumption was wrong, and the image of a continuous antipodal map must include the origin.

7 References

The proof of the Koebe embedding theorem presented here is due to Thurston. The version that was presented appeared in the book Combinatorial Geometry, by Janos Pach and Pankaj K. Agarwal.

The proof of Theorem 2 appears in the paper "Spectral Partitioning Works: planar graphs and finite-element meshes" by Spielman and Teng. A short version appeared in FOCS '96. The full version may be found at "http://theory.lcs.mit.edu/~spielman/Berk/spect.html".

8 Acknowledgements

The circle packing diagrams in these notes were made using Oliver Goodman's *Mathematica* package `CirclePack`. It is available from *http://www.geom.umn.edu/software/download/circlepack.html.*