Spectral Graph Theory and its Applications	Lecture 20
Graph Decompositions	
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20.1 Graph Decomposition

I am now going to prove that every graph has a good decomposition. I'll begin by defining what I mean by decomposition. First, let's recall the definition of the conductance of a graph from Lecture 7.

For a partition of the vertex set of a graph $(S, \overline{S})$, we define the conductance of the cut to be

$$\Phi(S) \stackrel{\text{def}}{=} \frac{\sum_{u \in S, v \notin S} a_{u,v}}{\min\left(\sum_{w \in S} d_w, \sum_{w \notin S} d_w\right)}.$$

To simplify writing expressions such as this, we define the volume of a set of vertices S by

$$\text{vol}(S) \stackrel{\text{def}}{=} \sum_{w \in S} d_w,$$

the volume of a set of edges F to be

$$\text{vol}(F) \stackrel{\text{def}}{=} \sum_{(u,v) \in F} a_{u,v},$$

and

$$\partial(S) \stackrel{\text{def}}{=} \left\{(u,v) \in E : u \in S, v \in \overline{S}\right\}.$$

So, we can write

$$\Phi(S) = \frac{\text{vol}(\partial(S))}{\min(\text{vol}(S), \text{vol}(\overline{S}))}.$$

The conductance of a graph is given by

$$\Phi(G) \stackrel{\text{def}}{=} \min_{S \subset V} \Phi(S).$$

I will define a ϕ-decomposition of a graph $G = (V, E)$ to be a partition of V into sets $V_1, \ldots, V_k$ such that for all i, the graph G_i induced on V_i satisfies $\Phi(G_i) \geq \phi$. Note that $G_i = (V_i, E_I)$ where E_i is the set of edges both of whose endpoints lie in V_i. The boundary of a decomposition, $\partial(V_1, \ldots, V_k)$ is the set of edges going between components of the partition: $E - \cup_i E_i$.

Our first goal will be to prove that there exist $(1/\log n)$-decompositions with $|\partial(V_1, \ldots, V_n)| \leq |E|/2$. We will prove this by first cutting G, then cutting each of the two halves, and so on. So that we can be sure this process does not go on for too long, we will force our cuts to be balanced. We will show that, if they are not balanced, then it is not necessary to keep cutting the larger side.

When we cut V into two pieces, S and $\overline{S}$, we will consider the induced graphs $G(S)$ and $G(\overline{S})$. Accordingly, we define $\mathrm{vol}_{V-S}(T)$ and $\partial_{V-S}(T)$ to be the volume and boundary of the set T in the graph $G(\overline{S})$.

We note that the volume of a set T in G is different from its volume in $G(S)$, as the latter could be smaller:

Proposition 20.1.1. *Let $S \subseteq V$ and $T \subseteq \overline{S}$. Then,*

$$\mathrm{vol}_{V-S}(T) \leq \mathrm{vol}_V(T).$$

As we will need to consider cuts in $\overline{S}$, we also observe:

Proposition 20.1.2. *Let $S \subset V$ and $T \subset \overline{S}$. Then*

$$\partial(S \cup T) = \partial_V(S) + \partial_{V-S}(T).$$

Lemma 20.1.3. *For any $\phi \leq \Phi(G)$, let S be the largest set of size at most $|V|/2$ such that $\phi(S) \leq \phi$. If $|S| < |V|/4$, then*

$$\Phi(G(\overline{S})) \geq \phi/3.$$

Proof. Assume by way of contradiction that $\Phi(G(\overline{S})) < \phi/3$. Then, there exists a set $R \subseteq \overline{S}$ such that

$$\Phi_{V-S}(R) = \frac{\partial_{V-S}(R)}{\mathrm{vol}_{V-S}(R).}$$

We then let T be the set R or $V - S - R$ such that $\mathrm{vol}_V(T) \leq \mathrm{vol}_V(V-S)/2$, and note that

$$\frac{\partial_{V-S}(T)}{\mathrm{vol}_V(T)} < \phi/3.$$

We will now obtain a contraction by showing that one of $S \cup T$ or $\overline{S \cup T}$ would have been chosen instead of S.

We first consider the case in which $\mathrm{vol}(S \cup T) \leq \mathrm{vol}(V)/2$. In this case, we have As $\mathrm{vol}(S \cup T) \leq \mathrm{vol}(V)/2$, we know that

$$\begin{aligned}
\phi(S \cup T) &= \frac{\partial(S \cup T)}{\mathrm{vol}(S \cup T)} \\
&\leq \frac{\partial(S) + \partial_{V-S}(T)}{\mathrm{vol}(S \cup T)} \\
&= \frac{\partial(S) + \partial_{V-S}(T)}{\mathrm{vol}(S) + \mathrm{vol}(T)} \\
&\leq \max\left(\frac{\partial(S)}{\mathrm{vol}(S)}, \frac{\partial_{V-S}(T)}{\mathrm{vol}(T)}\right)
\end{aligned}$$

by my favorite inequality. Finally, both terms in the max are less than ϕ, providing the contradiction.

We now consider the case in which $\mathrm{vol}(S \cup T) > \mathrm{vol}(V)/2$. In this case, we need to show that $\mathrm{vol}(S \cup T)$ can not be too big. We know that $\mathrm{vol}(T) \leq (\mathrm{vol}(V) - \mathrm{vol}(S))/2$, so

$$\mathrm{vol}(S \cup T) = \mathrm{vol}(S) + \mathrm{vol}(T) \leq \frac{\mathrm{vol}(V) - \mathrm{vol}(S)}{2} + \mathrm{vol}(S) \leq \frac{\mathrm{vol}(V) + \mathrm{vol}(S)}{2} < (5/8)\mathrm{vol}(V).$$

So, $\mathrm{vol}(\overline{S \cup T}) \geq (3/8)\mathrm{vol}(V)$. One the other hand,

$$\begin{aligned}
\partial(S \cup T) &= \partial(S) + \partial_{V-S}(T) \\
&\leq \phi\mathrm{vol}(S) + (\phi/3)\mathrm{vol}(T) \\
&\leq \phi\mathrm{vol}(S) + (\phi/3)\left(\mathrm{vol}(V) - \mathrm{vol}(S)\right)/2 \\
&\leq (5/6)\phi\mathrm{vol}(S) + (1/6)\phi(\mathrm{vol}(V)) \\
&< (3/8)\phi(\mathrm{vol}(V)).
\end{aligned}$$

Thus,

$$\phi(\overline{S \cup T}) = \frac{\partial(S \cup T)}{\mathrm{vol}(\overline{S \cup T})} < \frac{(3/8)\phi\mathrm{vol}(V)}{(3/8)\mathrm{vol}(V)} < \phi,$$

which is the contradiction we were looking for. □

We can now state and analyze our decomposition procedure. Given a graph G, we will find the most balanced set S such that $\Phi(S) \leq \phi$, and take this as our cut. If $\mathrm{vol}(S) < \mathrm{vol}(V)/4$, then we know that $\Phi(\overline{S}) \geq \phi/3$, and so we will not have to consider this set, and we can just recurse on S. Otherwise, we recurse on both sets. Thus, the depth of our recursion is at most

$$\log_{4/3}(\mathrm{vol}(V)).$$

As the number of edges cut at each stage of the recursion is at most a $\phi/2$ fraction of the volume, we find that the total number of edges cut is at most

$$\frac{\mathrm{vol}(V)\log_{4/3}(\mathrm{vol}(V))\phi}{2}.$$

If we now set $\phi = 1/2\log_{4/3}(\mathrm{vol}(V))$, then at most half the edges will be cut, and when the procedure ends, each set will have conductance at least $\phi/3 = 1/6\log_{3/2}(\mathrm{vol}(V))$.