I have tried to arrange these problems in order of difficulty. They can all be solved using material covered in the lectures so far.

1. Let $G$ be a $d$-regular bipartite graph, and let $A$ be its adjacency matrix.
   a. Prove that $-d$ is an eigenvalue of $A$, and find the corresponding eigenvector.
   b. Prove that for every eigenvalue $\mu$ of $A$, $-\mu$ is also an eigenvalue.

2. Let $A$ be a non-negative symmetric matrix. Let $d_i$ be the sum of the entries in the $i$th row of $A$, and let $D = \text{diag}(d_1, \ldots, d_n)$. Let $S = D^{-1}(A + D)/2$. We will consider multiplying $S$ by vectors on the right, that is $Sx$. Note that random walks multiply by this matrix on the left.
   a. Prove that $S1 = 1$.
   b. Prove that for every non-negative vector $x$,
      \[ \max_i x(i) \geq \max_i (Sx)(i). \]
   c. Removed from Problem Set

3. A $(d, c)$-extremely regular graph is a connected $d$-regular graph in which every pair of vertices has exactly $c$ common neighbors. (we do not consider a vertex to be a neighbor of itself)
   a. Let $A$ be the adjacency matrix of an extremely regular graph. Prove that $A$ has at most two distinct eigenvalues.
   b. Let $A$ be the adjacency matrix a regular graph. Prove that if $A$ has at most two distinct eigenvalues, then $A$ is the complete graph. (Hint: consider $A = VDV^T$)
4. Let $A$ be the adjacency matrix of a connected weighted graph.
   
a. Prove that $A$ has an eigenvector with positive entries. (Hint: note that $A$ and $A^k$ have the same eigenvectors)

   b. Let $\mu$ be the eigenvalue of that positive eigenvector. Prove that every other eigenvalue is smaller in absolute value. (Hint: for any other eigenvector $(x_1, \ldots, x_n)$, consider $(|x_1|, |x_2|, \ldots, |x_n|)$.)

5. Let $G = (V, W, E)$ be a connected $d$-regular bipartite graph and let $A$ be its adjacency matrix. Assume that every eigenvalue of $A$ other than $d$ and $-d$ has absolute value at most $\mu$. Let $S \subseteq V$ and $T \subseteq W$, and let $e(S, T)$ denote the number of edges between $S$ and $T$. Prove that

$$e(S, T) \leq \frac{2d |S| |T|}{|S| + |T|} + \mu n.$$ 

Hint: this generalizes a consequence of Theorem 9.2.1.