Spectral Graph Theory and its Applications **out:** October 18, 2004

Problem Set 2

Lecturer: Daniel A. Spielman **due:** November 9, 2004

I have tried to arrange these problems in order of difficulty. They can all be solved using material covered in the lectures so far.

1. Let G be a d-regular bipartite graph, and let A be its adjacency matrix.

   a. Prove that $-d$ is an eigenvalue of A, and find the corresponding eigenvector.

   b. Prove that for every eigenvalue μ of A, $-\mu$ is also an eigenvalue.

2. Let A be a non-negative symmetric matrix. Let d_i be the sum of the entries in the ith row of A, and let $D = \texttt{diag}(d_1, \ldots, d_n)$. Let $S = D^{-1}(A + D)/2$. We will consider multiplying S by vectors on the right, that is Sx. Note that random walks multiply by this matrix on the left.

   a. Prove that $S\mathbf{1} = \mathbf{1}$.

   b. Prove that for every non-negative vector x,

   $$\max_i x(i) \geq \max_i (Sx)(i).$$

   c. **Removed from Problem Set**

3. A (d, c)-*extremely regular graph* is a connected d-regular graph in which every pair of vertices has exactly c common neighbors. (we do not consider a vertex to be a neighbor of itself)

   a. Let A be the adjacency matrix of an extremely regular graph. Prove that A has at most two distinct eigenvalues.

   b. Let A be the adjacency matrix a regular graph. Prove that if A has at most two distinct eigenvalues, then A is the complete graph. (Hint: consider $A = VDV^T$)

4. Let A be a the adjacency matrix of a connected weighted graph.

    a. Prove that A has an eigenvector with positive entries. (Hint: note that A and A^k have the same eigenvectors)

    b. Let μ be the eigenvalue of that positive eigenvector. Prove that every other eigenvalue is smaller in absolute value. (Hint: for any other eigenvector $(x_1, \ldots, x_n)$, consider $(|x_1|, |x_2|, \ldots, |x_n|)$).

5. Let $G = (V, W, E)$ be a connected d-regular bipartite graph and let A be its adjacency matrix. Assume that every eigenvalue of A other than d and $-d$ has absolute value at most μ. Let $S \subseteq V$ and $T \subseteq W$, and let $e(S, T)$ denote the number of edges between S and T. Prove that

$$e(S,T) \leq \frac{2d\,|S|\,|T|}{|S| + |T|} + \mu n.$$

    Hint: this generalizes a consequence of Theorem 9.2.1.