On Contact Points of Convex Bodies [*]

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Abstract

We show that for every convex body K in $\mathbb{R}^n$, there is a convex body H such that

$$H \subset K \subset c \cdot H \qquad \text{with } c = 2.24$$

and H has at most $O(n)$ contact points with the minimal volume ellipsoid that contains it. When K is symmetric, we can obtain the same conclusion for every constant $c > 1$. We build on work of Rudelson [5], who showed the existence of H with $O(n \log n)$ contact points. The approximating body H is constructed using the 'barrier' method of Batson, Spielman, and the author, which allows one to extract a small set of vectors with desirable spectral properties from any John's decomposition of the identity. The main technical contribution of this paper is a way of controlling the *mean* of the vectors produced by that method, which is necessary in the application to John's decompositions of nonsymmetric bodies.

1 Introduction

Let K be an arbitary convex body in $\mathbb{R}^n$ and let $\mathcal{E}$ be a minimal volume ellipsoid containing K. Then the *contact points* of K are the points of intersection of $\mathcal{E}$ and K. The ellipsoid $\mathcal{E}$ is unique and characterized by a celebrated theorem of F. John [1], which says that if K is embedded via an affine transformation in $\mathbb{R}^n$ so that $\mathcal{E}$ becomes the standard Euclidean ball B_2^n, then there are $m \leq N = n(n+3)/2$ contact points $x_1, \ldots, x_m \in K \cap B_2^n$ and nonnegative weights $c_1, \ldots c_m$ for which

$$\sum_i c_i x_i = 0 \qquad \text{(mean zero)} \qquad (1)$$

$$\sum_i c_i x_i x_i^T = I \qquad \text{(inertia matrix identity).} \qquad (2)$$

Moreover, any convex body K' containing $x_1, \ldots, x_m$ must have B_2^n inside its John ellipsoid. We refer to a weighted collection of unit vectors $(c_i, x_i)_{i \leq m}$ which satisfies (1) and (2) as a *John's decomposition* of the identity.

The study of contact points has been fruitful in Convex Geometry, for instance in understanding the behavior of volume ratios of symmetric and nonsymmetric convex bodies [1] and in estimating distances between convex bodies and the cube or simplex [5, 3]. In this work, we consider the

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number of contact points of a convex body. Define a distance d between two (not necessarily symmetric) convex bodies K and H in $\mathbb{R}^n$ as follows[1]:

$$d(K,H) = \inf_{T \in GL(n), u,v \in \mathbb{R}^n} \{c : H + u \subset T(K+v) \subset c(H+u)\}$$

and let $\mathcal{K}$ be the space of all convex bodies equipped with the topology induced by d. Gruber [4] proved that the set of K having fewer than $N = n(n+3)/2$ contact points is of the first Baire category in $\mathcal{K}$. However, Rudelson has shown that every K is arbitrarily close to a body which has a much smaller number of contact points.

Theorem 1 (Rudelson [6])**.** *Suppose K is a convex body in $\mathbb{R}^n$ and $\epsilon > 0$. Then there is a convex body H such that $d(H,K) \le 1+\epsilon$ and H has at most $m \le Cn\log n/\epsilon^2$ contact points, where C is a universal constant.*

In this note, we show that the $\log n$ factor in Rudelson's theorem is unnecessary in many cases. For symmetric convex bodies, we obtain exactly the same distance guarantee $d(H,K) \le 1+\epsilon$ but with a much smaller number $m \le 32n/\epsilon^2$ of contact points of H. For arbitrary convex bodies, we show a somewhat weaker result that only guarantees an H within constant distance $d(H,K) \le 2.24$, with $m \le Cn$ contact points for some universal C. Thus Rudelson's $O(n\log n)$ bound is still the best known in the regime $d(H,K) < 2.24$ for nonsymmetric bodies.

Our approach for constructing H is the same as Rudelson's, and consists of two steps:

1. Given a John's decomposition $(c_i, x_i)_{i \le m}$ for K, extract a small subsequence of points x_i which are *approximately* a John's decomposition. To be precise, find a set of scalars b_i, at most $s = O(n)$ of which are nonzero, and a small 'recentering' vector u for which

$$\sum_i b_i(x_i+u) = 0 \qquad \left\| I - \sum_i b_i(x_i+u)(x_i+u)^T \right\| \le \epsilon.$$

2. Use the approximate John's decomposition $(b_i, x_i+u)_{i \le s}$ to construct an *exact* John's decomposition $(a_i, u_i)_{i \le s}$, and show that it characterizes the John Ellipsoid of a body H that is close to K.

The source of our improvement is a new method for extracting the approximate decomposition in step (1). Whereas the b_i were chosen by random methods in Rudelson's work, we now use a deterministic procedure which was introduced in recent work of Batson et al. on spectral sparsification of graphs [2]. The main theorem of [2] is the following:

Theorem 2 (Batson, Spielman, Srivastava)**.** *Suppose $d > 1$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are vectors in $\mathbb{R}^n$ with*

$$\sum_{i \le m} \mathbf{v}_i \mathbf{v}_i^T = I.$$

Then there exist scalars $s_i \ge 0$ with $|\{i : s_i \ne 0\}| \le dn$ so that

$$I \preceq \sum_{i \le m} s_i \mathbf{v}_i \mathbf{v}_i^T \preceq \left(\frac{\sqrt{d}+1}{\sqrt{d}-1} \right)^2 I.$$

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[1] When K and H are symmetric then we can take $u = v = 0$ and d becomes the usual Banach-Mazur distance.

A sharp result regarding the contact points of *symmetric* convex bodies can be derived as an immediate corollary of Theorem 2 and Rudelson's proof of Theorem 1.1 [5].

Corollary 3. *If K is a symmetric convex body in $\mathbb{R}^n$ and $\epsilon > 0$, then there exists a body H such that $H \subset K \subset (1+\epsilon)H$ and H has at most $m \leq 32n/\epsilon^2$ contact points with its John Ellipsoid.*

Proof. Suppose K is a symmetric convex body whose John ellipsoid is B_2^n, and let $X = \{x_1, \dots x_m\}$ be contact points satisfying (1,2) with weights $c_1, \dots c_m$. Since K is symmetric we can assume that $x_i \in X \iff -x_i \in X$, and that the corresponding weights c_i are equal.

We will extract an approximate John's decomposition from X. Apply Theorem 2 to the vectors $\mathbf{v}_i = \sqrt{c_i} x_i$ with parameter $d = 16/\epsilon^2$ to obtain scalars s_i, and let $Y \subset X$ be the set of x_i with nonzero s_i. We are now guaranteed that

$$I \preceq \sum_{x_i \in Y} s_i c_i x_i x_i^T \preceq \left(\frac{4/\epsilon + 1}{4/\epsilon - 1} \right)^2 I$$

with $|Y| \leq 16n/\epsilon^2$. Notice that by an easy calculation

$$\left(\frac{4/\epsilon + 1}{4/\epsilon - 1} \right)^2 \leq 1 + \epsilon$$

for sufficiently small epsilon.

In order to obtain a John's decomposition from these vectors, we need to ensure the mean zero condition (1). This is achieved easily by taking a negative copy of each vector in Y and halving the scalars s_i, since

$$\sum_{x_i \in Y} (s_i/2) c_i x_i + (s_i/2) c_i (-x_i) = 0$$

and

$$\sum_{x_i \in Y} (s_i/2) c_i x_i x_i^T + (s_i/2) c_i (-x_i)(-x_i)^T = \sum_{x_i \in Y} s_i c_i x_i x_i^T$$

which we know is a good approximation to the identity. Thus the vectors in $Y \cup -Y$ with weights $b_i = s_i c_i / 2$ on x_i and $-x_i$ constitute a $(1+\epsilon)$-approximate John's decomposition with only $32n/\epsilon^2$ points. Substituting this fact in place of [5, Lemma 3.1] in the proof of [5, Theorem 1.1] gives the promised result. $\square$

When the body K is not symmetric, there is no immediate way to guarantee the mean zero condition. If we simply recenter the vectors produced by Theorem 2 to have mean zero by adding $u = -\frac{\sum_i b_i x_i}{\sum_i b_i}$ to each x_i, then the corresponding inertia matrix is

$$\sum_i b_i (x_i + u)(x_i + u)^T = \sum_i b_i x_i x_i^T - \left(\sum_i b_i \right) u u^T \tag{3}$$

which no longer well-approximates the identity if $\| \left(\sum_i b_i \right) u u^T \|$ is large. This is the issue that we address here. In Section 2, we prove a variant of Theorem 2 which allows us to obtain very good control on the mean u at the cost of having a worse (at best factor 4, rather than $1+\epsilon$) approximation of the inertia matrix to the identity. In Section 3, we show that this is still sufficient to carry out Rudelson's construction of the approximating body H. The end result is the following theorem.

Theorem 4. *For every $\epsilon > 0$ the following is true for n sufficiently large. If K is a convex body in $\mathbb{R}^n$, then there is a convex body H such that*

$$H \subset K \subset (\sqrt{5} + \epsilon)H$$

and H has at most $O_\epsilon(n)$ contact points with its John Ellipsoid.

2 Approximate John's Decompositions

In this section we will prove the following theorem.

Theorem 5. *Suppose we are given a John's decomposition of the identity, i.e., unit vectors $x_1, \ldots, x_m \in \mathbb{R}^n$ with nonnegative scalars c_i such that*

$$\sum_i c_i x_i = 0 \tag{4}$$

$$\sum_i c_i x_i x_i^T = I. \tag{5}$$

Then for every $\epsilon > 0$ there are scalars b_i, at most $O_\epsilon(n)$ nonzero, and a vector u such that

$$I \preceq \sum_i b_i (x_i + u)(x_i + u)^T \preceq (4 + \epsilon) I \tag{6}$$

$$\sum_i b_i (x_i + u) = 0 \tag{7}$$

$$\left(\sum_i b_i \right) \|u\|^2 \leq \epsilon. \tag{8}$$

We remark that the requirement (4) is necessary to allow a useful bound on u, since otherwise we can take $x_i = e_i$ with $c_i = 1$ for the canonical basis vectors $\{e_i\}_{i \leq n}$ and it is easily checked (for instance, using concavity of λ_{min})) that

$$\sum_i b_i e_i e_i^T - \frac{\left(\sum_i b_i e_i\right)\left(\sum_i b_i e_i\right)}{\sum_i b_i}$$

is singular for every choice of scalars b_i, which is worthless considering (3).

2.1 An outline of the proof

As in the proof of Theorem 2 [2], we will build the approximate John's decomposition $(b_i, x_i + u)$ by an iterative process which adds one vector at a time. At any step of the process, let

$$A = \sum_j b_j x_j x_j^T$$

denote the inertia matrix of the vectors that have already been added (i.e., the b_i's that have been set to some nonzero value), and let

$$z = \sum_j b_j x_j$$

denote their weighted sum. Initially both $A = 0$ and $z = 0$. We will take $s = O(n)$ steps, adding one $b_j v_j v_j^T$ in each step, in a way that guarantees that at the end the inertia matrix A approximates the identity and the sum z is close to zero. Since we only increment one b_i in each step, at most $O(n)$ will be nonzero at the end, as promised.

Barrier Functions. The choice of vector to add in each step will be guided by two 'barrier' potential functions which we will use to maintain control on the eigenvalues of A. For real numbers $u, l \in \mathbb{R}$, which we will call the upper and lower barrier respectively, define:

$$\Phi^u(A) \stackrel{\text{def}}{=} \operatorname{Tr}(uI - A)^{-1} = \sum_i \frac{1}{u - \lambda_i} \quad \text{(Upper potential).}$$

$$\Phi_l(A) \stackrel{\text{def}}{=} \operatorname{Tr}(A - lI)^{-1} = \sum_i \frac{1}{\lambda_i - l} \quad \text{(Lower potential),}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

As long as $A \prec uI$ and $A \succ lI$ (i.e., $\lambda_{\max}(A) < u$ and $\lambda_{\min}(A) > l$), these potential functions measure how far the eigenvalues of A are from the barriers u and l. In particular, they blow up as any eigenvalue approaches a barrier, since then $uI - A$ (or $A - lI$) approaches a singular matrix. Thus if $\Phi_l(A)$ and $\Phi^u(A)$ are appropriately bounded, we can conclude that the eigenvalues of A are 'well-behaved' in that there is no accumulation near the barriers u and l. This will allow us to prove that the process never gets stuck.

For a thorough discussion of these potential functions, where they come from, and why they work, see [2].

Invariants. We will maintain three invariants throughout the process. Note that u, l, A and z vary from step to step, while P_U, P_L, and ϵ remain fixed.

- The eigenvalues of A lie strictly between l and u:

$$lI \prec A \prec lU. \tag{9}$$

- Both the upper and lower potentials are bounded by some fixed values P_L and P_U:

$$\Phi_l(A) \leq P_L \qquad \Phi^u(A) \leq P_U. \tag{10}$$

- The running sum z is appropriately small:

$$\|z\|^2 \leq \epsilon \cdot \operatorname{Tr}(A) \tag{11}$$

Initialization. At the beginning of the process we have $A = 0$ and $z = 0$, and the barriers at initial values

$$l = l_0 = -1 \quad \text{and} \quad u = u_0 = 1. \tag{12}$$

It is easy to see that (9), (10), and (11) all hold with $P_U = P_L = n$ at this point.

Maintenance. The process will evolve in steps. Each step will consist of adding two vectors, tv and rw, where $t, r \geq 0$ and $v, w \in \{x_i\}_{i \leq m}$. We will call these the main vector and the fix vector, respectively.

The main vector will allow us to move the upper and lower barriers forward by fixed amounts $\delta_l > 0$ and $\delta_u > 0$ while maintaining the invariants (9) and (10); in particular, we will choose it in a manner which satisfies:

$$\Phi_{l+\delta_l}(A+tvv^T) \le \Phi_l(A) \qquad \Phi^{u+\delta_u}(A+tvv^T) \le \Phi^u(A) \qquad (l+\delta_l)I \prec A+tvv^T \prec (u+\delta_u)I. \tag{13}$$

The fix will correct any undesirable impact that the main has on the sum; specifically, we will choose rw in a way that guarantees

$$\langle z, tv+rw\rangle \le 0 \tag{14}$$

where z is the sum at the end of the previous step. Thus the net increase in the length of the sum in any step is given by

$$\|z+(tv+rw)\|^2 = \|z\|^2 + 2\langle z, tv+rw\rangle + \|tv+rw\|^2 \le \|z\|^2 + (t+r)^2, \tag{15}$$

since v and w are unit vectors. The corresponding increase in the trace is simply $t+r$, and so if we guarantee in addition that the steps are sufficiently small:

$$t+r \le \epsilon \tag{16}$$

then the invariant (11) can be maintained by induction as follows:

$$\begin{aligned} \frac{\|z+(tv+rw)\|^2}{\operatorname{Tr}(A+tvv^T+rww^T)} &\le \frac{\|z\|^2+(t+r)^2}{\operatorname{Tr}(A)+(t+r)} && \text{by (15)} \\ &\le \max\left\{\frac{\|z\|^2}{\operatorname{Tr}(A)}, \frac{(t+r)^2}{t+r}\right\} \\ &\le \epsilon && \text{by (16).} \end{aligned}$$

However, we need to make sure that adding the fix vector does not cause us to violate (9) or (10). To do this, the addition of rw will be accompanied by an appropriately large shift of $\delta_u^f > 0$ in the upper barrier. In particular, we will make sure that on top of satisfying (14), rw also satisfies

$$\Phi^{u+\delta_u+\delta_u^f}(A+tvv^T+rww^T) \le \Phi^{u+\delta_u}(A+tvv^T) \quad \text{and} \quad A+tvv^T+rww^T \prec (u+\delta_u+\delta_u^f)I. \tag{17}$$

Since $A+tvv^T+A+rww^T \succeq A+tvv^T$, the analogous bound for the lower potential follows immediately without any additional lower shift:

$$\Phi_{l+\delta_l+0}(A+tvv^T+rww^T) \le \Phi_{l+\delta_l}(A+tvv^T) \quad \text{with} \quad A+tvv^T+rww^T \succ (l+\delta_l)I.$$

Together with (13), these inequalities guarantee that

$$\Phi_{l+\delta_l}(A+tvv^T+rww^T) \le \Phi_l(A) \le P_L$$

and

$$\Phi^{u+\delta_u+\delta_u^f}(A+tvv^T+rww^T) \le \Phi^u(A) \le P_U,$$

thus maintaining both (9) and (10), as desired.

To summarize what has occured during the step: we have added two vectors tv and rw and shifted u and l forward by $\delta_u+\delta_u^f$ and δ_l, respectively, in a manner that our three invariants continue

to hold. To show that such a step can actually be taken, we need to prove that as long as the invariants are maintained there must exist scalars $t, r \geq 0$ and vectors $v, w \in \{x_i\}_{i\leq m}$ which satisfy all of the conditions (13), (14), (16), and (17). We will do this in Lemma 6.
Termination. After s steps of the process, we have

$$(-1+s\delta_l)I \prec A \prec (1+s(\delta_u+\delta_u^f))I \qquad \text{by (9).}$$

If we take s sufficiently large, then we can make the ratio $\lambda_{\max}(A)/\lambda_{\min}(A)$ arbitrarily close to $\frac{\delta_u+\delta_u^f}{\delta_l}$. In the actual proof, which we will present shortly, we will show that the process can be realized with parameters $\delta_l, \delta_u, \delta_u^f$ for which this ratio can be made arbitrarily close to 4 in $s = O(n)$ steps.

As for the mean, we will set $u = -\frac{\sum_j b_j x_j}{\sum_j b_j} = -\frac{z}{\mathrm{Tr}(A)}$ at the end of the process, so that immediately

$$(\sum_j b_j)\|u\|^2 = \frac{\|z\|^2}{\mathrm{Tr}(A)} \leq \epsilon \tag{18}$$

by (11) as desired.

2.2 Realizing the proof

To complete the proof, we will identify a range of parameters $\delta_l, \delta_u, \delta_u^f, P_U, P_L$, and ϵ for which the above process can actually be sustained.

Lemma 6 (One Step). *Suppose $(c_i, x_i)_{i\leq m}$ is a John's decomposition and z is any vector. Let $A \succeq 0$ be a matrix satisfying the invariants (9) and (10). If*

$$\frac{1}{\delta_u} + \frac{1}{\delta_u^f} + 2P_U + P_L + \frac{4n}{\epsilon} \leq \frac{1}{\delta_l} \tag{19}$$

Then there are scalars $t, r \geq 0$ and vectors $v, w \in \{x_i\}$ which satisfy (13), (14), (16), and (17).

To this end, we recall the following lemmas from [2], which characterize how much of a vector one can add to a matrix without increasing the upper and lower potentials.

Lemma 7 (Upper Barrier Shift). *Suppose $A \prec uI$ and $\delta_u > 0$. Then there is a positive definite matrix $\mathbb{U} = \mathbb{U}(A, u, \delta_u)$ so that if $\mathbf{v}$ is any vector which satisfies*

$$\mathbf{v}^T\mathbb{U}\mathbf{v} \geq \frac{1}{t}$$

then

$$\Phi^{u+\delta_u}(A + t\mathbf{v}\mathbf{v}^T) \leq \Phi^u(A) \quad \text{and } \lambda_{\max}(A + t\mathbf{v}\mathbf{v}^T) < u + \delta_u.$$

That is, if we add t times $\mathbf{v}\mathbf{v}^T$ to A and shift the upper barrier by δ_u, then we do not increase the upper potential.

Lemma 8 (Lower Barrier Shift). *Suppose $A \succ lI$, $\delta_l > 0$, and $\Phi_l(A) < 1/\delta_l$. Then there is a matrix $\mathbb{L} = \mathbb{L}(A, l, \delta_l)$ so that if $\mathbf{v}$ is any vector which satisfies*

$$\mathbf{v}^T\mathbb{L}\mathbf{v} \leq \frac{1}{t}$$

then

$$\Phi_{l+\delta_l}(A + t\mathbf{v}\mathbf{v}^T) \le \Phi_l(A) \quad \textit{and} \quad \lambda_{\min}(A + t\mathbf{v}\mathbf{v}^T) > l + \delta_l.$$

That is, if we add t times $\mathbf{v}\mathbf{v}^T$ to A and shift the lower barrier by δ_l, then we do not increase the lower potential.

We will prove that desirable vectors exist by taking averages of the quantities $\mathbf{v}^T\mathbb{U}\mathbf{v}$ and $\mathbf{v}^T\mathbb{L}\mathbf{v}$ over our contact points $\{x_i\}$ with weights $\{c_i\}$. Since

$$\begin{aligned}\sum_i c_i x_i^T \mathbb{U} x_i &= \mathbb{U} \bullet \left(\sum_i c_i x_i x_i^T\right) \\ &= \mathbb{U} \bullet I && \text{by (2)} \\ &= \mathrm{Tr}(\mathbb{U})\end{aligned}$$

(and similarly for $\mathbb{L}$), it will be useful to recall bounds on $\mathrm{Tr}(\mathbb{U})$ and $\mathrm{Tr}(\mathbb{L})$ from [2]. Crucially, these bounds do not depend on the matrix A or on u, l at all, but only on the shifts δ_u and δ_l and on the potentials.

Lemma 9 (Traces of $\mathbb{L}$ and $\mathbb{U}$)**.** *If $lI \prec A \prec uL$ with $\Phi_l(A) \le P_L$ and $\Phi^u(A) \le P_U$ then*

$$\mathrm{Tr}(\mathbb{U}) \le \frac{1}{\delta_u} + P_U$$

and

$$\mathrm{Tr}(\mathbb{L}) \ge \frac{1}{\delta_l} - P_L.$$

We are now in a position to prove Lemma 6.

Proof of Lemma 6. Let $\mathbb{L} = \mathbb{L}(A, l, \delta_l), \mathbb{U} = \mathbb{U}(A, u, \delta_u), \mathbb{U}^f = \mathbb{U}(A, u + \delta_u, \delta_u^f)$ be the matrices produced by lemmas 7 and 8.

Let us focus on the main vector first. By Lemmas 7 and 8, we can add tv without increasing potentials if

$$v^T\mathbb{U}v \le 1/t \le v^T\mathbb{L}v.$$

In fact, we will insist on v for which

$$v^T\mathbb{U}v + 2/\epsilon \le 1/t \le v^T\mathbb{L}v$$

as this will ensure that we can take $t \le \epsilon/2$.

Let $D(v) = v^T\mathbb{L}v - v^T\mathbb{U}v - 2/\epsilon$ and call $\mathcal{F} = \{x_i : D(x_i) \ge 0\}$ the set of *feasible* vectors. Let $\mathcal{P} = \{x_i : \langle x_i, z\rangle > 0\}$ be the set of vectors with positive inner product with z, and let $\mathcal{N} = \{x_i : \langle x_i, z\rangle \le 0\}$ be the vectors in the complementary halfspace.

We will always add as little of a main vector can, so we can assume that we take $1/t = v^T\mathbb{L}v$ whenever $v \in \mathcal{F}$. Here is the rule for choosing which v to add: choose the feasible v for which $t\langle v, z\rangle$ is minimized. If this quantity is negative then there is no need for a fix vector, and taking $w = 0$ we are done. Otherwise let $\alpha := \min\{t\langle v, z\rangle : v \in \mathcal{F}\}$ and notice that $\mathcal{F} \subset \mathcal{P}$:j

$$\frac{\langle v, z\rangle}{\alpha} \ge \frac{1}{t} = v^T\mathbb{L}v \quad \forall v \in \mathcal{F}. \tag{20}$$

Taking a sum, we find that

$$\begin{aligned}
\sum_{\mathcal{P}} c_i \frac{\langle x_i, z\rangle}{\alpha} &\geq \sum_{\mathcal{F}} c_i \frac{\langle x_i, z\rangle}{\alpha} && \text{since } \mathcal{F} \subset \mathcal{P} \\
&\geq \sum_{\mathcal{F}} c_i x_i^T \mathbb{L} x_i && \text{by (20)} \\
&\geq \sum_{\mathcal{F}} c_i D(x_i) && \text{since } \mathbb{U} \succeq 0 \text{ implies that } D(x_i) \leq x_i^T \mathbb{L} x_i \\
&\geq \sum_i c_i D(x_i) && \text{since } D < 0 \text{ outside } \mathcal{F}
\end{aligned}$$

However, since $\sum_i c_i x_i = 0$ this implies that

$$\sum_{\mathcal{N}} c_i \frac{\langle x_i, -z\rangle}{\alpha} = \sum_{\mathcal{P}} c_i \frac{\langle x_i, z\rangle}{\alpha} \geq \sum_i c_i D(x_i). \tag{21}$$

We will use (21) to show that a suitable fix vector w exists. We are interested in finding a $w \in \{x_i\}$ and $r \geq 0$ for which

$$\begin{aligned}
r\langle w, -z\rangle &\geq \alpha && \text{(sufficient to reverse } \alpha = t\langle v, z\rangle\text{)—for (14)} \\
w^T \mathbb{U}^f w + 2/\epsilon &\leq 1/r && \text{(upper barrier feasible with shift } \delta_u^f \text{ and } r \leq \epsilon/2\text{)—for (16,17)}
\end{aligned}$$

Thus it suffices to find a w for which

$$w^T \mathbb{U}^f w + 2/\epsilon \leq \frac{\langle w, -z\rangle}{\alpha},$$

and then we can squeeze $1/r$ in between. Taking a weighted sum over all vectors of interest, it will be sufficient to show that

$$\sum_{\mathcal{N}} c_i x_i^T \mathbb{U}^f x_i + 2c_i/\epsilon \leq \sum_{\mathcal{N}} c_i \frac{\langle x_i, -z\rangle}{\alpha}.$$

For the left hand side we use the crude estimate

$$\sum_{\mathcal{N}} c_i x_i^T \mathbb{U}^f x_i + 2c_i/\epsilon \leq \sum_{\mathcal{N}\cup\mathcal{P}} c_i x_i^T \mathbb{U}^f x_i + 2c_i/\epsilon = \mathrm{Tr}(\mathbb{U}^f) + 2n/\epsilon$$

and for the right hand side we consider that

$$\begin{aligned}
\sum_{\mathcal{N}} c_i \frac{\langle x_i, -z\rangle}{\alpha} &\geq \sum_i c_i D(x_i) && \text{by (21)} \\
&= \sum_i c_i x_i^T \mathbb{L} x_i - c_i x_i^T \mathbb{U} x_i - 2c_i/\epsilon \\
&= \mathrm{Tr}(\mathbb{L}) - \mathrm{Tr}(\mathbb{U}) - 2n/\epsilon && \text{since } \sum_i c_i x_i x_i^T = I
\end{aligned}$$

Thus it will be enough to have

$$\mathrm{Tr}(\mathbb{U}^f) + 2n/\epsilon \leq \mathrm{Tr}(\mathbb{L}) - \mathrm{Tr}(\mathbb{U}) - 2n/\epsilon$$

which follows from our hypothesis (19) and Lemma 9. □

Proof of Theorem 5. If we start with $l_0 = -1$ and $u_0 = 1$ then we can take $P_L = P_U = n$. Setting $\delta_u = \delta_u^f = (2+\epsilon)\delta_l$, (19) reduces to

$$\frac{2}{(2+\epsilon)\delta_l} + 3n + \frac{4n}{\epsilon} \leq \frac{1}{\delta_l},$$

which it is easy to check is satisfied for small enough δ_l, in particular for

$$\delta_l = \frac{\epsilon^2}{10n}.$$

At the end of s steps we take

$$u = -\frac{\sum_i b_i x_i}{\sum_i b_i} = -\frac{z}{\mathrm{Tr}(A)},$$

immediately satisfying (7) and (8). To finish the proof, we notice that

$$(-1 + s\delta_l - \frac{\|z\|^2}{\mathrm{Tr}(A)})I \preceq \sum_i b_i(x_i + u)(x_i + u)^T = A - \mathrm{Tr}(A)uu^T \preceq (1 + s \cdot 2(2+\epsilon)\delta_l)I.$$

Setting $s = 100n/\epsilon^3$ and replacing ϵ by $\epsilon/3$ yields (6), as promised. □

3 Construction of the Approximating Body

The contents of this section are very similar to Rudelson [5, Section 4] but the calculations are more delicate because we must work with a $(4+\epsilon)-$approximate John's decomposition rather than a $(1+\epsilon)$-approximate one.

The main technical contribution is a generic procedure for turning approximate John's decompositions into approximating bodies:

Lemma 10. *Suppose K has contact points $(c_i, x_i)_{i\leq m}$ and $(b_i, y_i = x_i + u)_{i\leq s}$ are the vectors produced by Theorem 5, with*

$$A = \sum_i b_i y_i y_i^T$$

satisfying

$$\kappa(A) = \|A\| \cdot \|A^{-1}\| = O(1).$$

Then for n sufficiently large, there is a body H with at most s contact points and

$$d(H, K) \leq (1 + o(1)) \left(\kappa(A)\left(1 + \frac{(\sqrt{\kappa(A)} - 1)^2}{4}\right)\right)^{1/2}. \tag{22}$$

This immediately yields a proof of Theorem 4:

Proof of Theorem 4. Since the condition number $\kappa(A)$ guaranteed by Theorem 5 can be made arbitrarily close to 4, the number in (22) can be made arbitrarily close to

$$\left(4\left(1 + \frac{1}{4}\right)\right)^{1/2} = \sqrt{5}$$

as desired. □

Let $(b_i, y_i = x_i + u)_{i \le s}$ be the approximate John's decomposition guaranteed by Theorem 5, with

$$\sum_i b_i y_i = 0$$

and

$$\sum_i b_i y_i y_i^T = A.$$

There are two problems with this: the y_i are not unit vectors, and their moment ellipsoid $\mathcal{E} = A^{1/2} B_2^n$ is not the sphere. We will adjust the vectors in a manner that fixes both these problems, to obtain an exact John's decomposition $(\hat{a}_i, \hat{u}_i)_{i \le s}$.

Add a small vector v, to be determined later, to each y_i to obtain vectors

$$\hat{y}_i := y_i + v$$

with inertia matrix

$$\hat{A} := \sum_i b_i \hat{y}_i \hat{y}_i^T = \sum_i b_i y_i y_i^T + \sum_i b_i v v^T = A + \mathrm{Tr}(A) v v^T. \tag{23}$$

(we will use $\hat{}$ to denote vectors that depend on v.) Let $\hat{R} := \hat{A}^{1/2}$ and let $\hat{E} := \hat{R} B_2^n$ be the corresponding moment ellipsoid. If we rescale each $\hat{y}_i$ to lie on $\hat{E}$, taking

$$\hat{z}_i := \frac{\hat{y}_i}{\|\hat{y}_i\|_{\hat{E}}} \qquad \text{where } \|\hat{y}_i\|_{\hat{E}} = \|\hat{R}^{-1} \hat{y}_i\|, \tag{24}$$

and then apply the inverse transformation $\hat{R}^{-1}$ which maps $\hat{E}$ to B_2^n, we obtain unit vectors

$$\hat{u}_i := \hat{R}^{-1} \hat{z}_i.$$

Moreover, if these are given weights

$$\hat{a}_i := b_i \|\hat{y}_i\|_{\hat{E}}^2$$

then we have an exact decomposition of the identity since

$$\sum_i \hat{a}_i \hat{u}_i \hat{u}_i^T = \hat{R}^{-1} \left(\sum_i b_i \|\hat{y}_i\|_{\hat{E}}^2 \frac{\hat{y}_i \hat{y}_i^T}{\|\hat{y}_i\|_{\hat{E}}^2} \right) \hat{R}^{-1} = \hat{R}^{-1} \hat{A} \hat{R}^{-1} = I. \tag{25}$$

In the following lemma, we show that there must exist a small v for which the weighted sum

$$\sum_i \hat{a}_i \hat{u}_i = \sum_i b_i \|\hat{y}_i\|_{\hat{E}}^2 \hat{R}^{-1} \frac{\hat{y}_i}{\|\hat{y}_i\|_{\hat{E}}} = \hat{R}^{-1} \left(\sum_i b_i \|\hat{y}_i\|_{\hat{E}} \hat{y}_i \right)$$

is equal to zero. This will complete the construction of $(\hat{a}_i, \hat{u}_i)_{i \le s}$.

Lemma 11. *Let b_i, y_i, A, etc. be as above and suppose $\mathrm{Tr}(A) = \Omega(n)$. Then there is a vector v with*

$$\|v\| \le \nu_A := \frac{1 + o(1)}{2} \left(\sqrt{\kappa(A)} - 1 \right) \sqrt{\frac{\|A\|}{\mathrm{Tr}(A)}} \tag{26}$$

for which

$$\sum_i b_i \|\hat{y}_i\|_{\hat{E}} \hat{y}_i = 0.$$

Proof. We need to find a v for which

$$\sum_i b_i \sqrt{(y_i+v)^T \hat{A}^{-1}(y_i+v)}(y_i+v) = 0.$$

As in [5], we will do this using the Brouwer fixed point theorem. In particular it will suffice to show that the function

$$\begin{aligned} F(v) &= -\frac{\sum_i b_i \beta_i^{(v)} y_i}{\sum_i b_i \beta_i^{(v)}} && \text{where } \beta_i^{(v)} = \sqrt{(y_i+v)^T (A + \mathrm{Tr}(A) v v^T)^{-1} (y_i+v)} \\ &= -\frac{\sum_i b_i (\beta_i^{(v)} - \mu) y_i}{\sum_i b_i \beta_i^{(v)}} && \text{for any } \mu \in \mathbb{R} \text{ since } \sum_i b_i y_i = 0 \end{aligned}$$

maps $\nu_A B_2^n$ to itself. We begin with the preliminary bounds

$$\begin{aligned} \beta_i^{(0)} &= \sqrt{(x_i+u)^T A^{-1} (x_i+u)} \\ &\le \|x_i+u\| \sqrt{\|A^{-1}\|} \\ &\le (1+o(1)) \sqrt{\|A^{-1}\|} \quad \text{since } \|x_i\| = 1 \text{ and } \|u\| \le O(\mathrm{Tr}(A)^{-1/2}), \end{aligned} \tag{27}$$

and similarly

$$\beta_i^{(0)} \ge (1-o(1)) \frac{1}{\|A\|^{1/2}}, \tag{28}$$

for all i. We now have the estimate:

$$\begin{aligned} \|F(v)\| &= \max_{\|w\|=1} \frac{\sum_i b_i (\beta_i^{(v)} - \mu) \langle y_i, w\rangle}{\sum_i b_i \beta_i^{(v)}} \\ &\le \frac{(\sum_i b_i (\beta_i^{(v)} - \mu)^2)^{1/2}}{\sum_i b_i \beta_i^{(v)}} \cdot \max_{\|w\|=1} \left(\sum_i b_i \langle y_i, w\rangle^2 \right)^{1/2} && \text{by Cauchy-Schwarz} \\ &= \frac{(\sum_i b_i (\beta_i^{(v)} - \mu)^2)^{1/2}}{\sum_i b_i \beta_i^{(v)}} \cdot \|A\|^{1/2} \\ &\le \frac{1}{1-o(1)} \frac{(\sum_i b_i (\beta_i^{(v)} - \mu)^2)^{1/2}}{\sum_i b_i \beta_i^{(0)}} \cdot \|A\|^{1/2} && \text{by Lemma 12} \\ &\le (1+o(1)) \cdot \frac{(\sum_i b_i (\beta_i^{(v)} - \mu)^2)^{1/2}}{\sum_i b_i} \cdot \|A\|^{1/2} \cdot \|A\|^{1/2} && \text{by (28)} \end{aligned}$$

Applying Lemma 13, we control the sum in the numerator as

$$\begin{aligned}
(\sum_i b_i(\beta_i^{(v)} - \mu)^2)^{1/2} &\leq \left(\operatorname{Tr}(A)^{1/2} + \sum_i b_i(\beta_i^{(0)} - \mu)^2\right)^{1/2} \\
&\leq (\sum_i b_i(\beta_i^{(0)} - \mu)^2)^{1/2} + O(\operatorname{Tr}(A)^{1/4}) \qquad \text{using } \sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \\
&\leq \left(\sum_i b_i\right)^{1/2} \cdot \max_i |\beta_i^{(0)} - \mu| + O(\operatorname{Tr}(A)^{1/4}) \\
&= \operatorname{Tr}(A)^{1/2} \cdot \left(\frac{|\max_i \beta_i^{(0)} - \min_i \beta_i^{(0)}|}{2} + o(1)\right) \\
&\qquad \text{setting } \mu = (\max_i \beta_i^{(0)} - \min_i \beta_i^{(0)})/2 \\
&\leq (1 + o(1)) \cdot \frac{\operatorname{Tr}(A)^{1/2}}{2}\left(\|A^{-1}\|^{1/2} - \frac{1}{\|A\|^{1/2}}\right) \\
&= (1 + o(1) \cdot \frac{\operatorname{Tr}(A)^{1/2}}{2\|A\|^{1/2}}\left(\sqrt{\kappa(A)} - 1\right).
\end{aligned}$$

Substituting into the previous bound gives

$$\|F(v)\| \leq (1 + o(1)) \cdot \frac{\sqrt{\kappa(A)} - 1}{2} \cdot \sqrt{\frac{\|A\|}{\operatorname{Tr}(A)}},$$

as advertised in (26). ☐

Lemma 12. *If* $\|v\| = O(\operatorname{Tr}(A)^{-1/2})$*, then*

$$\sum_i b_i\beta_i^{(v)} \geq \sum_i b_i\beta_i^{(0)} - O(\operatorname{Tr}(A)^{1/2}) \geq (1 - o(1)) \cdot \sum_i b_i\beta_i^{(v)}.$$

Proof. We can lowerbound the individual terms as

$$\begin{aligned}
\beta_i^{(v)} &= \|\hat{R}^{-1}(y_i + v)\| \\
&\geq \|\hat{R}^{-1}y_i\| - \|\hat{R}^{-1}v\| \\
&= \left(y_i^T(A + \operatorname{Tr}(A)vv^T)^{-1}y_i\right)^{1/2} - \|\hat{R}^{-1}v\| \\
&= \left(y_i^T\left(A^{-1} - \frac{A^{-1}\operatorname{Tr}(A)vv^TA^{-1}}{1 + \operatorname{Tr}(A)v^TA^{-1}v}\right)y_i\right)^{1/2} - \|\hat{R}^{-1}v\| \qquad \text{by the Sherman-Morrison formula} \\
&\geq \sqrt{y_i^TA^{-1}y_i} - \left(y_i^T\left(\frac{A^{-1}vv^TA^{-1}}{\operatorname{Tr}(A)^{-1} + v^TA^{-1}v}\right)y_i\right)^{1/2} - \|\hat{R}^{-1}v\| \qquad \text{since } \sqrt{a-b} \geq \sqrt{a} - \sqrt{b}
\end{aligned}$$

Taking a sum, we observe the difference of the sums that we are interested in is bounded by

$$\sum_i b_i\beta_i^{(0)} - \sum_i b_i\beta_i^{(v)} \leq \sum_i b_i\left(y_i^T\left(\frac{A^{-1}vv^TA^{-1}}{\operatorname{Tr}(A)^{-1} + v^TA^{-1}v}\right)y_i\right)^{1/2} + \sum_i b_i\|\hat{R}^{-1}v\|. \tag{29}$$

The first sum is handled by Cauchy-Schwarz:

$$\begin{aligned}\sum_i b_i \left(y_i^T \left(\frac{A^{-1}vv^TA^{-1}}{\mathrm{Tr}(A)^{-1} + v^TA^{-1}v} \right) y_i \right)^{1/2} &\le \left(\sum_i b_i\right)^{1/2} \left(\sum_i b_i y_i^T \left(\frac{A^{-1}vv^TA^{-1}}{\mathrm{Tr}(A)^{-1} + v^TA^{-1}v} \right) y_i \right)^{1/2} \\ &= \mathrm{Tr}(A)^{1/2} \left(\frac{\mathrm{Tr}(AA^{-1}vv^TA^{-1})}{\mathrm{Tr}(A)^{-1} + v^TA^{-1}v} \right)^{1/2} \\ &\qquad \text{since} \sum_i b_i y_i y_i^T = A \\ &< \mathrm{Tr}(A)^{1/2}.\end{aligned}$$

For the second, we observe crudely that

$$\sum_i b_i \|\hat{R}^{-1}v\| \le \mathrm{Tr}(A) \|\hat{R}^{-1}\| \|v\| = O(\mathrm{Tr}(A)^{1/2})$$

since $\|\hat{R}^{-1}\| = O(1)$ and $\|v\| = O(\mathrm{Tr}(A)^{-1/2})$. Plugging these two bounds into (29), we obtain

$$\begin{aligned}\sum_i b_i \beta_i^{(v)} &\ge \sum_i b_i \beta_i^{(0)} - O(\mathrm{Tr}(A)^{1/2}) \\ &\ge \sum_i b_i \beta_i^{(0)} \left(1 - \frac{O(\mathrm{Tr}(A)^{1/2})}{(\min_i \beta_i^{(0)}) \cdot \sum_i b_i} \right) \\ &\ge (1 - o(1)) \sum_i b_i \beta_i^{(0)} \qquad \text{noting that } \min_i \beta_i^{(0)} = \Omega(1).\end{aligned}$$

☐

Lemma 13. *If* $\|v\| = O(\mathrm{Tr}(A)^{-1/2})$ *then*

$$\sum_i b_i (\beta_i^{(v)} - \mu)^2 \le \sum_i b_i (\beta_i^{(0)} - \mu)^2 + O(\mathrm{Tr}(A)^{1/2}).$$

Proof. Write

$$\sum_i b_i (\beta_i^{(v)} - \mu)^2 = \sum_i b_i (\beta_i^{(v)})^2 - 2b_i \beta_i^{(v)} + b_i \mu^2.$$

Then $(\beta_i^{(v)})^2 \le (\beta_i^{(0)})^2$ by $A + \mathrm{Tr}(A)vv^T \succeq A$, and

$$-2\sum_i b_i \beta_i^{(v)} \le -2 \sum_i b_i \beta_i^{(0)} + O(2\mathrm{Tr}(A)^{1/2}) \quad \text{by Lemma 12,}$$

as desired. ☐

Proof of Lemma 10. We are now in a position to construct the body H promised in Theorem 4. Let K be a convex body with B_2^n as its John ellipsoid and contact points $(c_i, x_i)_{i \le m}$. Use Theorem 5 to obtain a subsequence $(b_i, y_i = x_i + u)_{i \le s}$ with only $O(n)$ points. Let v, $\hat{A}$, $\{\hat{z}_i\}_{i \le s}$, $\{\hat{y}_i\}_{i \le s}$, $\hat{R}$, $\hat{E}$, and $(\hat{a}_i, \hat{u}_i)_{i \le s}$ be as above and take

$$\hat{K} = K + u + v.$$

Let

$$\theta_{min} := \min_{\|y\|_{\hat{E}}=1} \|y\|_2,$$

be the size of the largest copy of B_2^n which fits inside $\hat{E}$, let $\epsilon > 0$ be a small constant, and consider the body

$$H = \text{conv}\left(\frac{\theta_{min}}{1+\epsilon}\hat{K}, \hat{z}_1, \ldots, \hat{z}_s\right), \tag{30}$$

which is as a shrunken version of $\hat{K}$ with 'spikes' $\hat{z}_i$.

We first show that H has very few contact points with its John Ellipsoid, which is $\hat{E}$. Observe that each $\hat{z}_i \in \partial\hat{E}$ since $\|\hat{z}_i\|_{\hat{E}} = 1$ by construction. Moreover, as the John Ellipsoid of K is B_2^n we have the containments

$$\frac{\theta_{min}}{1+\epsilon}\hat{K} \subset \frac{\theta_{min}}{1+\epsilon}(B_2^n + u + v) \subset \frac{\theta_{min}(1+o(1))}{1+\epsilon}B_2^n,$$

since $\|u\|, \|v\| = O(n^{-1/2})$ and $\theta_{min} = \Omega(1)$, immediately implying that

$$\frac{\theta_{min}}{1+\epsilon}\hat{K} \in \text{int}(\theta_{min}B_2^n) \subset \text{int}(\hat{E}).$$

Thus we must have $\hat{z}_i \notin \text{conv}\frac{\theta_{min}}{1+\epsilon}\hat{K}$ and $\hat{z}_i \in \partial H$; moreover, these are the *only* contact points of H with $\hat{E}$. To see that $\hat{E}$ is indeed the John Ellipsoid of H, apply the inverse transformation $\hat{R}^{-1}$, and note that $\hat{R}^{-1}H$ has contact points $\hat{u}_i = \hat{R}^{-1}\hat{z}_i$ with B_2^n, which we have already shown satisfy the conditions of John's theorem with weights $\hat{a}_i$ in (25) and Lemma 11.

It remains to bound the distance $d(K, H)$; we will show that

$$\frac{\theta_{min}}{1+\epsilon}\hat{K} \subset H \subset (1+\epsilon)\theta_{max}\hat{K},$$

for $\theta_{max} := \max_{\|y\|_{\hat{E}}=1} \|y\|_2 = \max_{\|y\|=1} \|y\|_{\hat{E}}^{-1}$. The first containment is obvious; for the second, observe that for each i we have

$$\begin{aligned}
\|\hat{z}_i\|_{\hat{K}} &= \frac{\|x_i + u + v\|_{\hat{K}}}{\|\hat{y}_i\|_{\hat{E}}} \\
&= \frac{1}{\|\hat{y}_i\|_{\hat{E}}} \quad \text{since } \hat{y}_i = x_i + u + v \in \partial\hat{K} \\
&\\
&\le \|\hat{y}_i\| \cdot \theta_{max} \\
&\le (1+o(1))\theta_{max} \quad \text{since } \hat{y}_i = x_i + u + v \text{ and } \|u\|, \|v\| = O(n^{-1/2}).
\end{aligned}$$

The distance between H and K is thus bounded by the ratio

$$(1+\epsilon)^2\frac{\theta_{max}}{\theta_{min}} \le (1+\epsilon)^2\frac{\max_{\|y\|=1}\|y\|_{\hat{E}}}{\min_{\|y\|=1}\|y\|_{\hat{E}}} \le (1+\epsilon)^2\sqrt{\kappa(\hat{A})}. \tag{31}$$

To complete the proof, we bound $\kappa(\hat{A})$ using Theorem 5 (6) and Lemma 11 as follows:

$$\begin{aligned}
\kappa(\hat{A}) &= \|\hat{A}\|\|\hat{A}^{-1}\| \\
&\leq \|A + \mathrm{Tr}(A)vv^T\|\|A^{-1}\| \qquad \text{since } \hat{A} = A + \mathrm{Tr}(A)vv^T \succeq A \\
&\leq \left(\|A\| + \mathrm{Tr}(A)\|v\|^2\right)\|A^{-1}\| \\
&\leq \left(\|A\| + (\frac{1}{4} + o(1))(\sqrt{\kappa(A)} - 1)^2\|A\|\right)\|A^{-1}\| \qquad \text{by (26)} \\
&= \kappa(A)\left(1 + (\frac{1}{4} + o(1))(\sqrt{\kappa(A)} - 1)^2\right).
\end{aligned}$$

Combining this with (31) and taking $\epsilon = o(1)$ gives the required bound (22). □

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