

Shape from Shading on a Cloudy Day

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Abstract

In the classical Shape from Shading model, surface luminance depends primarily on the unit surface normal. However, under diffuse lighting conditions, such as the sky on a cloudy day, luminance depends primarily on the amount of sky visible from each surface element, with surface normal of secondary importance. This claim is formalized in terms of a DOMINATING SKY PRINCIPLE and a SURFACE APERTURE function. An approximately functional constraint between surface luminance and aperture emerges. It is shown how to use this constraint to efficiently recover a depth map from an image. A curious difference from the classical problem is uncovered. When one assumes a point light source, the local geometric constraints of the Shape from Shading problem lie along the surface. However, in the diffuse lighting problem, the local geometric constraints are found in a VISIBILITY FIELD which is defined in the free space above the surface.

1 Introduction

Shape from Shading has been defined as the problem of computing a depth map from a piecewise smooth intensity image [1]. Classical solutions to this problem are based on an assumption that surface luminance is determined by the surface normal. For outdoor scenes on a sunny day, when there is a well defined light source direction, this classical assumption is usually valid. However, in many other situations, the assumption fails. In particular, when a scene is illuminated by diffuse light source such as the sky on a cloudy day, surface luminance is not determined by the surface normal. This is the situation with which we are concerned.

To illustrate, consider a scene composed of a single convex object resting on a ground plane. It has been argued by a number of authors (eg. [2]) that, under diffuse lighting conditions, classical Shape from Shading techniques could be used to recover the shape of the object. The idea was that the region of the sky which directly illuminates a point on the convex object would be entirely determined by the surface normal at that point.

Observe however, that these techniques would fail to recover the shape of the ground plane. Since the surface normal of the ground plane is constant, the classical shading model requires that the luminance of the ground plane is also constant. But this requirement is clearly not met, since different points on the ground plane are exposed to different amounts of the diffuse light source. (In particular, points which are closer to the object receive more light from the diffuse source than points which are further from the object.)

A second example should clarify this limitation of the classical model. Figure 1 shows an artificial image formed by overlapping a set of rectangles. Each rectangle has a ramp intensity in the vertical direction. Most people perceive this image as depicting a scene composed of parallel flat surfaces (eg. a skyline or a graveyard). The percept is valid since, under diffuse lighting conditions, the luminance of such protruding surfaces would vary with height. Points which are higher along a given patch would receive more illumination from the diffuse source than points below.

Clearly, the percept of parallel flat surface patches is inconsistent with classical Shape from Shading which requires that smooth luminance variations are entirely due to surface normal variations. In the skyline interpretation, the surface normals are constant along each surface patch, yet the luminance is clearly not constant.

The classical Shape from Shading model cannot, in general, account for

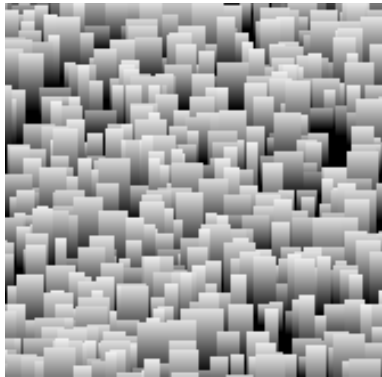
surface shading under diffuse lighting conditions. In this paper, we will present a new Shape from Shading model which is designed specifically for these conditions. The main idea of the model is that, under diffuse lighting conditions, surface luminance is determined primarily by the amount of the diffuse source to which a surface element is exposed. We will show how this model can be used to recover a depth map from a given image.

2 Overview

For a general scene, the relationship between surface material, surface geometry and surface luminance is governed by the Radiosity Equation ([3, 4]). This is an integral equation which explicitly relates the luminance of each surface point in a scene to the luminance of all the other surface points. In Section 3, we discuss the Radiosity Equation for a scene composed of Lambertian surfaces with constant albedo, and illuminated by a uniform diffuse light source.

In Section 4, we present a Dominating Sky Principle. This is an inequality which relates the luminance of surfaces in a scene to the brightness of the uniform diffuse light source. In Section 5 we introduce a useful geometric quantity, the surface aperture, which measures the solid angle of the light source that is visible from the surface. A second inequality emerges from this definition. Together, these two inequalities allow us to approximate the Radiosity Equation by expressing the surface luminance in terms of the surface aperture and the surface albedo.

In Section 6, we extend our discussion from the geometry of incident



light to the geometry of ambient light by introducing geometric constraints which relate how much of the light source is visible from various points in the free space above the surface. In Section 7, we show how these “visibility constraints” can be used to compute Shape from Shading. Finally, in Section 8, we present a number of experimental results.

3 The Radiosity Equation

As we discussed in the introduction, classical Shape from Shading models are often inappropriate for scenes illuminated by a diffuse light source. In this section, we develop a framework from which a more appropriate model can be derived.

Let the free space in a scene be denoted as a point set, \mathcal{F} . Light is both absorbed and reflected at the surfaces in the scene, that is, at the boundary of free space, $\partial\mathcal{F}$. By definition, $\partial\mathcal{F}$ does not include points at infinity. Assume that the surfaces are smooth, so that the *unit surface normal* is well defined,

$$\mathbf{N} : \partial\mathcal{F} \rightarrow \text{unit sphere} .$$

Thus, for each $\mathbf{x} \in \partial\mathcal{F}$, there is a tangent plane at \mathbf{x} , as well as a hemisphere of directions from which \mathbf{x} can receive illumination. Let $\mathcal{H}(\mathbf{x})$ denote this *unit hemisphere of illuminant directions*,

$$\mathcal{H}(\mathbf{x}) \equiv \{ \mathbf{L} : \mathbf{L} \in \text{unit sphere, and } \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) > 0 \},$$

so that $\mathcal{H}(\mathbf{x})$ is an open subset of the unit sphere.

Definition 1 *For any surface point $\mathbf{x} \in \partial\mathcal{F}$, the SURFACE VISIBILITY at \mathbf{x} is the set of incident directions $\mathcal{V}(\mathbf{x})$ in which the diffuse light source is visible from \mathbf{x} . Thus, the directions in which other surfaces in the scene are visible from \mathbf{x} is $\mathcal{V}_C(\mathbf{x}) = \mathcal{H}(\mathbf{x}) \setminus \mathcal{V}(\mathbf{x})$.*

Notice that, for any $\mathbf{x} \in \partial\mathcal{F}$, the set $\mathcal{V}(\mathbf{x})$ is an open subset of $\mathcal{H}(\mathbf{x})$. Intuitively, if the light source were visible from \mathbf{x} in direction \mathbf{L} , then there would exist a neighborhood of directions (containing \mathbf{L}) in which the source were also visible from \mathbf{x} .

The topology of $\mathcal{V}(\mathbf{x})$ could be complicated. For example, if there were objects floating in space, such as balloons or birds, then isolated regions of the sky would be occluded and $\mathcal{V}(\mathbf{x})$ would have holes. Another example is

that certain objects, such as the branches of a tree, could disconnect $\mathcal{V}(\mathbf{x})$ into a finite number of connected components.

It is essential to distinguish the light incident upon and reflected from a surface element. Let $I_{in}(\mathbf{x}, \mathbf{L})$ denote the *brightness* of the light ray incident at \mathbf{x} from direction \mathbf{L} . The units of brightness are $\text{lm}/\text{sr}\cdot\text{m}^2$ [17]. Let $I_{out}(\mathbf{x})$ denote the *luminance* of \mathbf{x} also with units $\text{lm}/\text{sr}\cdot\text{m}^2$ [5]. Because we are assuming the surfaces are Lambertian, $I_{out}(\mathbf{x})$ does not depend on the direction of reflection.

Let $d\Omega$ denote a small *solid angle*, that is, an small element of area on the hemisphere $\mathcal{H}(\mathbf{x})$. This solid angle is centered at direction $\mathbf{L} \in \mathcal{H}(\mathbf{x})$. Assuming the albedo, ρ , is constant throughout the scene, we integrate over all incident directions to obtain

$$I_{out}(\mathbf{x}) = \frac{\rho}{\pi} \int_{\mathcal{H}(\mathbf{x})} I_{in}(\mathbf{x}, \mathbf{L}) \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) d\Omega . \quad (1)$$

Next, assume that the diffuse light source has uniform brightness so for any $\mathbf{L} \in \mathcal{V}(\mathbf{x})$,

$$I_{in}(\mathbf{x}, \mathbf{L}) \equiv I_{\mathcal{D}}.$$

Here we are ignoring non-uniformities which often arise in nature. For example, on overcast days there is a brightness peak in the zenith direction [6] and on days when the cloud layer is thin there is a peak in the direction of the sun.

The uniform sky assumption permits a decomposition of Equation (1) as follows:

$$I_{out}(\mathbf{x}) = \frac{\rho}{\pi} I_{\mathcal{D}} \int_{\mathcal{V}(\mathbf{x})} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) d\Omega + \frac{\rho}{\pi} \int_{\mathcal{V}_C(\mathbf{x})} I_{in}(\mathbf{x}, \mathbf{L}) \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) d\Omega . \quad (2)$$

The first component is due to the direct illumination from the light source. The second component is due to the light reflected off other surfaces in the scene.

For each $\mathbf{L} \in \mathcal{V}_C(\mathbf{x})$, let $\Pi(\mathbf{x}, \mathbf{L})$ denote the surface point seen from surface element \mathbf{x} in direction \mathbf{L} . Thus, $I_{out}(\Pi(\mathbf{x}, \mathbf{L}))$ is the luminance of the point $\Pi(\mathbf{x}, \mathbf{L})$. Neglect any light scattering within the free space, so that for any $\mathbf{x} \in \partial\mathcal{F}$ and for any $\mathbf{L} \in \mathcal{V}_C(\mathbf{x})$,

$$I_{in}(\mathbf{x}, \mathbf{L}) = I_{out}(\Pi(\mathbf{x}, \mathbf{L})). \quad (3)$$

The above assumptions can be summarize by the following model.

The Radiosity Equation:

$$I_{out}(\mathbf{x}) = \frac{\rho}{\pi} I_{\mathcal{D}} \int_{\mathcal{V}(\mathbf{x})} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) d\Omega + \frac{\rho}{\pi} \int_{\mathcal{V}_C(\mathbf{x})} I_{out}(\Pi(\mathbf{x}, \mathbf{L})) \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) d\Omega . \quad (4)$$

Three shading factors are explicitly represented in this equation. Mutual shadowing is represented by the partition of the integral into two domains. Mutual illumination is represented by the non-zero luminances within the second integral. The classical dependence on surface normal also plays a role.

Notice that, if $\mathcal{V}(\mathbf{x})$ were independent of \mathbf{x} , then the above model would reduce to the classical one in which an “equivalent” point source is well defined. As we have argued however, under diffuse lighting conditions, $\mathcal{V}(\mathbf{x})$ often depends quite strongly on \mathbf{x} .

Computational techniques for solving the Radiosity Equation (under far more general conditions) have been developed within the computer graphics community (eg. [7]). Their goal was to accurately render a scene with known geometry and known surface material. Surface geometry is approximated as a mesh of N planar surface elements, and the Radiosity Equation is approximated as a set of N linear equations.

A related technique has been applied to the classical Shape from Shading problem [8]. The strategy was to estimate surface shape using classical techniques (that is, ignoring mutual illumination and mutual shadowing), and then to iteratively adjust the computed surface until its luminance was both consistent with original image and satisfied the Radiosity Equation. It was argued that this method would converge provided that the initial estimate of the surface was sufficiently accurate, that is, provided that the classical shading effects dominated.

In the present paper, we develop a technique for estimating Shape from Shading under diffuse lighting conditions. Our image formation model is quite different from the classical model in that we assume luminance is determined primarily by shadowing rather than by the surface normal. In the next two sections, we will show how to obtain this model by an approximation of the Radiosity Equation.

4 The Dominating Sky Principle

Our discussion of shading under diffuse lighting conditions begins with a few familiar observations. When an object such as a tree or a bird is viewed against the sky, the silhouette of the object is clearly demarcated, although the luminance variations within the silhouette may be difficult to discern. Similarly, at dawn or dusk, many objects appear only as silhouettes. The reason for this phenomenon is that, under diffuse lighting conditions, the sky is typically much brighter than the ground. In particular, the following can be shown (see Appendix A for a proof):

Lemma 1 (The Dominating Sky Principle) *Let a scene be illuminated by a diffuse light source with uniform brightness, $I_{\mathcal{D}}$, and let the surfaces be Lambertian with constant albedo, ρ . Then the luminance of the surface is bounded as follows:*

$$0 \leq I_{out}(\mathbf{x}) \leq \rho I_{\mathcal{D}} .$$

In one large compilation of natural scenes [10], the mean albedo was found to be approximately 0.15 [11]. This data is consistent with a separate study in which measurements of ambient light were made over a large number of natural habitats[12]. The latter indicate that on overcast days the brightness of the sky is often 10 times as great as the brightness of the ground.

Many insect species provide wonderful support for the Dominating Sky Principle in the form of a mechanism that has evolved for “detecting the horizon” [13]. It has been shown that certain insects cannot stabilize their roll¹ when they are placed in a uniformly lit environment, that is, one which does not contain a horizon. Such insects are believed to stabilize their flight by visually controlling the attitude of their body with respect to the visible sky, thus maximizing the difference in illumination above and below them.

5 Surface Aperture and Mutual Shadowing

The Dominating Sky Principle implies that there is a relationship between the luminance of a surface element and the amount of the diffuse source that is visible from that element [9]. If the entire source were visible then the luminance would be maximal, whereas if very little of the source were visible

¹the rotation about the longitudinal axis of the body

then the luminance would be relatively low. A simple example should clarify this idea.

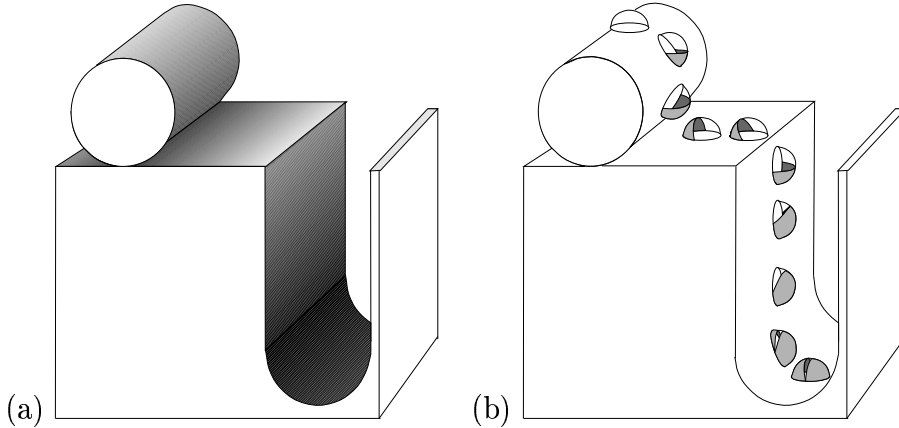


Figure 1: The hemisphere of incident directions above a point on a surface can be partitioned into two sets. $\mathcal{V}(\mathbf{x})$ denotes the set of directions in which the light source is visible from \mathbf{x} . Its complement, $\mathcal{V}_c(x)$, denotes the set of directions pointing to other surfaces in the scene.

Figure 1(a) shows a scene consisting of a cylinder lying next to a deep gully. The brightest region of the scene is the line on top of the cylinder, since from there the entire diffuse source is visible. The luminance decreases continuously as one moves to the bottom of the cylinder. From there, none of the diffuse source is visible. Similarly, as one descends into the gully, the diffuse source gradually becomes occluded, and the surface becomes darker. This shadowing effect, that different regions of the diffuse source are visible from different points on the surfaces, is represented geometrically in Figure 1(b).

In this section, we address the question of how much of the luminance variation along a surface is due to this shadowing effect. The following definition allows us to specify the degree of shadowing.

Definition 2 *The SURFACE APERTURE at $\mathbf{x} \in \partial\mathcal{F}$ is the percentage of incident directions in which the hemispheric source is visible. Formally,*

$$A(\mathbf{x}) \equiv \frac{1}{2\pi} \int_{\mathcal{V}(\mathbf{x})} d\Omega \ .$$

For the scene in Figure 1, the aperture would be 1 along the top of the cylinder, and 0 along the bottom of the cylinder. Aperture would increase as one were to move along the ground plane and away from the cylinder, and would decrease as one were to descend along the wall of the gully.

A useful bound on surface aperture is the following:

Lemma 2

$$A(\mathbf{x}) \leq \frac{1}{\pi} \int_{\mathcal{V}(\mathbf{x})} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) d\Omega \leq A(\mathbf{x})(2 - A(\mathbf{x})). \quad (5)$$

These bounds are calculated in Appendix B. The idea is to fix $A(\mathbf{x})$ and to consider what $\mathcal{V}(\mathbf{x})$ would have to be to maximize or minimize the given integral. For a maximum, $\mathcal{V}(\mathbf{x})$ would be set of directions bounded by a cone centered on $\mathbf{N}(\mathbf{x})$. For the minimum, $\mathcal{V}(\mathbf{x})$ would be an annulus of directions crowded around the horizon of $\mathcal{H}(\mathbf{x})$.

The Radiosity Equation and the Dominating Sky Principle may be combined to obtain

$$\frac{\rho}{\pi} I_{\mathcal{D}} \int_{\mathcal{V}(\mathbf{x})} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) d\Omega \leq I_{out}(\mathbf{x}) \leq \rho^2 I_{\mathcal{D}} + (1-\rho) \frac{\rho}{\pi} I_{\mathcal{D}} \int_{\mathcal{V}(\mathbf{x})} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) d\Omega. \quad (6)$$

Substituting (5) into (6) yields

$$\rho I_{\mathcal{D}} A(\mathbf{x})^2 \leq I_{out}(\mathbf{x}) \leq \rho^2 I_{\mathcal{D}} + (1-\rho) \rho I_{\mathcal{D}} A(\mathbf{x}) (2 - A(\mathbf{x})), \quad (7)$$

which can be rewritten as follows:

Proposition 1 (The Aperture-Luminance Inequality)

$$\max \left\{ 0, 1 - \sqrt{\frac{1}{1-\rho} \left(1 - \frac{I_{out}(\mathbf{x})}{\rho I_{\mathcal{D}}}\right)} \right\} \leq A(\mathbf{x}) \leq \sqrt{\frac{I_{out}(\mathbf{x})}{\rho I_{\mathcal{D}}}}. \quad (8)$$

How tight are these bounds of Equation (8) in practice ? To answer this question, we examine the values of $I_{out}(\mathbf{x})$ and $A(\mathbf{x})$ for a number of surfaces. Figure 2 shows a mesh plot of a drapery surface whose aperture values are well distributed over the range of 0 to 1. This surface was rendered with 40

point sources (an approximation of a uniform diffuse light source) using the Radiosity Equation².

Notice that the bounds of (8) are quite loose relative to the plotted values of $I_{out}(\mathbf{x})$ and $A(\mathbf{x})$. In particular, the data points are clustered near the middle of the allowed range. This clustering was expected. For a surface point to attain either the upper (or lower) bound of Equation (8), it is necessary that the upper (or lower) bounds of Equations (5) and (6) be attained *simultaneously*. This latter requirement is rarely met for two reasons. First, (5) and (6) represent distinct constraints, so an attainment of the bounds of one implies nothing about the attainment of the bounds of the other. Second, as is discussed in Appendices A and B, the conditions under which each of the bounds of Equations (5) and (6) are attained are rarely met.

To carry this investigation one step further, we examine a well known situation that is problematic for classical Shape from Shading. Concave surfaces which have large albedo typically do not satisfy the classical shading model, since the mutual illumination effect for such surfaces is large [15, 16]. Does a similar difficulty arise under uniform hemispheric lighting conditions?

We address this question by rendering a smooth concave surface, again using the Radiosity Equation. The depth map is a 50 X 50 concave Gaussian function having standard deviation 15 pixels and depth ranging from 0 to 50. In Figure 3, rendered images are shown for three different values of ρ , along with scatter plots of the $(I_{out}(\mathbf{x}), A(\mathbf{x}))$ pairs.

Clearly, the albedo affects the *mean* value of $A(\mathbf{x})$ for a given value of $I_{out}(\mathbf{x})$. That is, when ρ is large, the values of $I_{out}(\mathbf{x})$ are skewed toward larger luminances. However, the *variance* of $A(\mathbf{x})$ for a given value of $I_{out}(\mathbf{x})$ is roughly the same for all three albedos. The same qualitative observation can be made from the scatter plots of Figure 2.

The data suggest that the effect of mutual illumination is to skew the surface luminance distribution. In particular, as the albedo of a surface is increased, the distribution is skewed toward larger values. This result should be familiar to anyone who has skied. On a cloudy day, the snow covered hills (which have an albedo close to 1) have little contrast since the luminance distribution is greatly skewed toward a maximal value. As a result, the hills appear flat. This issue will be discussed again Section 8.1, when various choices of albedo are considered for a given image.

²Our rendering algorithm differs from the one described in [7]. Details will be presented in a later paper.

The above experiments suggest that surface aperture can be estimated by considering the average of the bounds of (8). Let $\tilde{A}(\mathbf{x})$ denote this estimate, so that

$$\tilde{A}(\mathbf{x}) \equiv \frac{1}{2} \left(\sqrt{\frac{I_{out}(\mathbf{x})}{\rho I_{\mathcal{D}}}} + \max \left\{ 0, 1 - \sqrt{\frac{1}{1-\rho} \left(1 - \frac{I_{out}(\mathbf{x})}{\rho I_{\mathcal{D}}} \right)} \right\} \right). \quad (9)$$

To summarize, under uniform diffuse lighting conditions, the luminance of a surface depends quite strongly on the surface aperture, that is, on the mutual shadowing of surfaces. This dependence is quite different from classical Shape from Shading models, where it is assumed that surface luminance is determined primarily by the surface normal.

6 The Visibility Field

Thus far, we have developed three functions: visibility, $\mathcal{V}(\mathbf{x})$, luminance, $I_{out}(\mathbf{x})$, and aperture, $A(\mathbf{x})$. Each of these was defined along the surfaces in a scene. We now extend the domains of $\mathcal{V}(\mathbf{x})$ and $A(\mathbf{x})$ to the free space, \mathcal{F} . That is, we extend our discussion from the geometry of incident light to the geometry of ambient light. This extension is useful because there are strong local constraints on the geometry of ambient light.

Definition 3 For any $\mathbf{x} \in \mathcal{F}$, let $\mathcal{V}(\mathbf{x})$ be the set of unit directions in which the sky is visible from \mathbf{x} . We refer to $\mathcal{V}(\mathbf{x})$ as the VISIBILITY FIELD at \mathbf{x} .

There are strong local constraints on the visibility field. Intuitively, for any $\mathbf{x} \in \mathcal{F}$, if the diffuse light source were visible in a particular direction, \mathbf{L} , then any translation within \mathcal{F} and parallel to \mathbf{L} would not affect this visibility. This constraint is formally expressed as follows:

The Local Visibility Constraint: Let $\mathbf{x} \in \mathcal{F}$ and $\epsilon > 0$ be such that the open ball centered at \mathbf{x} and of radius ϵ is entirely contained in \mathcal{F} . For any direction \mathbf{L} and for any $\delta \in [-\epsilon, \epsilon]$, we have $\mathbf{L} \in \mathcal{V}(\mathbf{x})$ if and only if $\mathbf{L} \in \mathcal{V}(\mathbf{x} + \delta \mathbf{L})$.

The Local Visibility Constraint allows us to compute Shape from Shading. The main concept (which will be developed in the next section) is that if the visibility field were known in a neighborhood within free space, then the Local Visibility Constraint could be used to locally expand this neighborhood.

The visibility field is related to a classical construction in illumination engineering [17]. The *light field* is the vector field on \mathcal{F} defined by

$$\mathbf{D}(\mathbf{x}) \equiv - \int_{\mathcal{V}(\mathbf{x})} I_{\mathcal{D}} \mathbf{L} d\Omega .$$

This vector field, $\mathbf{D}(\mathbf{x})$, is physically meaningful. It is the flux density of radiant energy coming directly from the diffuse light source. Although we will not make use of the light field when computing Shape from Shading, we will make use of the following scalar field.

Definition 4 For any $\mathbf{x} \in \mathcal{F}$, the APERTURE FIELD at \mathbf{x} is the percentage of the hemispheric source which is visible from \mathbf{x} :

$$A(\mathbf{x}) \equiv \frac{1}{2\pi} \int_{\mathcal{V}(\mathbf{x})} d\Omega .$$

Both the visibility and aperture fields will play an important role in our Shape from Shading computation. It is this computation to which we now turn.

7 The Computational Problem and Solution

A viewer is faced with the following computational problem. Given an image, compute a depth map which is consistent with the image, where consistency is defined with respect to Equation (9). There are two stages to the solution. The first is to transform the image into an estimate of the surface aperture function. The second stage is to use this estimated aperture function to compute the depth map of a surface.

7.1 Definition of Quantized Variables

Since digital images will be used as input, it is necessary to quantize the variables of the problem. We will often use the superscript, *, to remind the reader that a variable has been quantized.

Let \mathcal{Z} be the set of integers. We begin by quantizing space using a cubic lattice of nodes,

$$\mathcal{Z}^3 \equiv \{ \mathbf{x} = (x, y, n) : x, y, n \in \mathcal{Z} \}.$$

The set of *free nodes* in a scene is denoted as \mathcal{F}^* . These are the nodes lying strictly above the surfaces. The surface nodes are then denoted by $\partial\mathcal{F}^*$.

Assume that the image occupies a small solid angle of view, so that perspective distortions can be ignored. That is, the image projection is *orthographic*. Let $(0, 0, 1)$ be the *viewing direction*. \mathcal{P} is an $N \times N$ *lattice of pixels*, so that a depth map is a function

$$z : \mathcal{P} \rightarrow \{0, 1, \dots\}.$$

A *viewed surface* is thus a set of nodes,

$$\partial\mathcal{F}^* \equiv \{ (x, y, z(x, y)) : (x, y) \in \mathcal{P} \} \subset \mathcal{Z}^3.$$

Because of the limited field of view, it is necessary to make an assumption about the *image boundary conditions*. Assume that beyond the image window the surface has constant depth. Intuitively, the viewer is looking directly down at a pit which has been excavated from the ground. Further assume that the boundary depth is identical to the depth of the node nearest to viewer.

The quantized hemisphere of possible light source directions is denoted

$$\mathcal{H}^* \equiv \{ \mathbf{L}^k : k = 1, \dots, M \}.$$

These are the M directions above the ground plane from which a surface node can potentially be illuminated. \mathcal{H}^* is defined such that each direction is a unit vector pointing from the origin to a node in \mathcal{Z}^3 . Figure 4 depicts two quantizations of \mathcal{H}^* with $M = 32$ and 64 .

Definition 5 For any node $\mathbf{x} \in \mathcal{F}^* \cup \partial\mathcal{F}^*$, the QUANTIZED VISIBILITY FIELD at \mathbf{x} , $\mathcal{V}^*(\mathbf{x}) \subseteq \mathcal{H}^*$, is the finite set of directions in which the sky is visible from \mathbf{x} .

For any finite set, U , the number of elements of U is denoted $|U|$.

Definition 6 For any node $\mathbf{x} \in \mathcal{F}^* \cup \partial\mathcal{F}^*$, the QUANTIZED APERTURE FIELD is

$$A^*(\mathbf{x}) \equiv \frac{|\mathcal{V}^*(\mathbf{x})|}{|\mathcal{H}^*|}.$$

Finally, for any node $\mathbf{x} = (x, y, n)$, denote

$$\mathcal{V}_n^*(x, y) \equiv \mathcal{V}^*(\mathbf{x}), \quad A_n^*(x, y) \equiv A^*(\mathbf{x}).$$

7.2 From Image Brightness to Surface Aperture

Let $I^*(x, y)$ be the quantized image brightness, with units $\text{lm}/\text{sr}\cdot\text{m}^2$. In particular, the maximum image brightness is denoted I_{max}^* . According to the assumed image boundary conditions, the brightest surface node has aperture equal to 1. Hence, the following estimate is obtained from Equation (9):

$$\tilde{A}^*(\mathbf{x}) \equiv \frac{1}{2} \left(\sqrt{\frac{I_{out}(\mathbf{x})}{I_{max}^*}} + \max \left\{ 0, 1 - \sqrt{\frac{1}{1-\rho} \left(1 - \frac{I_{out}(\mathbf{x})}{I_{max}^*} \right)} \right\} \right). \quad (10)$$

Notice that it is necessary to choose particular value of ρ . This choice should be familiar to the reader, as it is also required in the classical Shape from Shading problem. As we will illustrate in Section 8.1, although the amplitude of the recovered surface varies with the error between the actual ρ and the chosen ρ , the location of the hills and valleys of the surface is approximately preserved.

7.3 From Surface Aperture to Depth

The second stage of the computation is to recover a depth map from the estimated surface aperture function. The algorithm is iterative, beginning at the depth of the node nearest to the viewer. The main idea is that, if the visibility field were known up to depth n , then it could be extended to depth $n+1$ using the following analog of the Local Visibility Constraints discussed in Section 6.

The Quantized Local Visibility Constraints: Let $\mathbf{x} \in \mathcal{F}^* \cup \partial\mathcal{F}^*$, and $\mathbf{L} \in \mathcal{H}^*$. If $\mathbf{x}' \in \mathcal{Z}^3$ is the nearest node to \mathbf{x} in direction \mathbf{L} , then $\mathbf{L} \in \mathcal{V}^*(\mathbf{x})$ if and only if all the following hold:

1. $\mathbf{x}' \in \mathcal{F}^*$, that is, \mathbf{x}' lies strictly above the surface;
2. $\mathbf{L} \in \mathcal{V}^*(\mathbf{x}')$;
3. any node on the open line segment between \mathbf{x} and \mathbf{x}' belongs to \mathcal{F}^* .

It is crucial to appreciate that these are local constraints. For each node at fixed depth, the directions in which the diffuse source is visible from that node can be computed by examining a small neighborhood above that node. The computation begins at the depth of the node nearest to the viewer. For all nodes at this depth, the entire source is assumed to be visible. Then, for

any given depth, n , and for any node $(x, y, n) \in \mathcal{F}^* \cup \partial\mathcal{F}^*$, the visibility field at (x, y, n) is computed using the Quantized Local Visibility Constraints. If the aperture of (x, y, n) were larger than the estimated aperture value of the corresponding pixel, then it would follow that $(x, y, n) \in \mathcal{F}^*$. For pixel (x, y) , the surface is reached when the computed aperture has decreased to the estimated aperture.

ALGORITHM: given an estimated aperture function, \tilde{A}^* , compute a depth map, z .

$\forall (x, y) \in \mathcal{P}, \quad z(x, y) := 0 ;$

$\forall (x, y) \in \mathcal{P}, \quad \forall n \leq 0, \quad \mathcal{V}_n^*(x, y) := \mathcal{H}^* ;$

$n := 0 ;$

REPEAT

$\forall (x, y) \in \mathcal{P},$

IF $z(x, y) = n$

 Compute $\mathcal{V}_n^*(x, y)$ using the Local Visibility Constraints;

 Compute $A_n^*(x, y)$ using Equation (6);

IF $A_n^*(x, y) > \tilde{A}^*(x, y)$

$z(x, y) := z(x, y) + 1;$

$n := n + 1;$

UNTIL $\forall (x, y) \in \mathcal{P}, \quad A_n^*(x, y) \leq \tilde{A}^*(x, y) \quad \square$

We have proved a number of results concerning the behavior of the algorithm[18].

Proposition 2 *The algorithm always converges.*

Proof : Because of the image boundary conditions, all nodes beyond a certain depth, z_{max} , would have minimum aperture. For points beyond this depth, the quantized diffuse source would not be visible. It follows that the algorithm could not proceed beyond z_{max} iterations. \square

Proposition 3 *Suppose that, for a given an aperture function A^* , there exists a depth map which is consistent with A^* . Then, the algorithm computes the shallowest depth map which is consistent with A^* .*

To summarize this section, the problem of computing Shape from Shading has been partitioned into two problems. First, a surface aperture function is estimated from a given image, using an image formation model based entirely on mutual shadowing. Second, a depth map of a surface having that estimated aperture function is computed.

8 Experimental Results

We now address three questions concerning the performance our algorithm.

8.1 The Choice of Albedo

In the first step of the algorithm, the surface aperture function is estimated. This estimate depends on a chosen albedo, ρ . The following experiment illustrates how the computed surface varies with the chosen ρ .

Consider again the depth map of Figure 2, and the rendered images of Figure 2. Our Shape from Shading algorithm is applied to each of these images, using three different albedos for each image. The resulting nine depth maps are shown in Figure 6, and the mean squared errors between the original depth map and the computed depth maps are shown in Table 1. In general, one would expect that the errors are smallest along the diagonal, that is, when the albedo is chosen correctly. Of course, since Equation (9) is an estimate, cases may arise where an incorrectly chosen albedo produces a smaller error. Indeed, for the example shown, the diagonal elements do not always have the smallest errors.

ρ	0.2	0.5	0.8
0.2	64.6	16.5	30.0
0.5	99.0	8.8	22.5
0.8	144.8	7.0	10.2

Table 1: This table contains the mean squared depth error of the computed surfaces of Fig.8. Each column corresponds to a single rendered image. Each row corresponds to a single chosen value of ρ . In general, one would expect that the errors are smallest along the diagonal (when the albedo is chosen correctly). Of course, since equation (6) is an estimate, it may happen that an incorrectly chosen albedo produces a smaller error. Indeed, for the example shown, the diagonal elements do not always have the smallest error.

When the chosen albedo is larger than the actual one, the reconstructed depth map has greater amplitude of variation than the actual one. The opposite effect occurs when the chosen albedo is too small. To appreciate why this occurs, recall how the rendered images of Figures 2 and 3 varied with the actual albedo. As the albedo was increased, the surface luminance becomes

skewed toward a maximal value. Thus, if the albedo were overestimated, then the image brightnesses would be less skewed than one would expect. This wider distribution of brightnesses would then be accounted for by a greater amplitude of variation in the computed depth map. Similarly, if the albedo were underestimated, then the skewing in the actual image brightnesses would be larger than expected. (That is, the image contrast would be smaller than expected.) Again, this unexpected brightness distribution would be accounted for by the computed depth map. It may be possible to use the distribution of image brightnesses in choosing the albedo of a given image. We are currently investigating this idea.

8.2 Quantization Errors

Although our mathematical results hold in principle, quantization effects are notorious, and require careful investigation. In the next experiment, seven constant depth maps were created. For each depth map, the aperture function was computed (see Appendix C). Then, a depth map was computed from each of these aperture functions. If the two computations (aperture from depth, and depth from aperture) were inverses of one another, then their successive application would reproduce the original constant depth maps. However, this is not the case. These errors in the computed depth map are entirely due to quantization.

Figure 7 shows the computed depth maps corresponding to $z \equiv 40$ for the coarse and fine quantization respectively. The symmetry of the error surface reflects the quantization of \mathcal{H}^* .

In Figure (??Figures lost??) the mean and standard deviation of the difference between the original constant depth map and the computed depth map are plotted for both a coarse and fine quantization. Notice first that the errors are smaller for the finer quantization, although not nearly fine enough to be ignored. Second, the errors are quite biased. The computed depth map is much shallower than the original. (This bias illustrates the effect mentioned previously in Proposition 3.) Third, the errors are greater for deeper depth maps. The reason is that, as the surfaces become deeper, the aperture becomes small. In this case, the diffuse source effectively becomes a local source, such as a window or a lamp. Clearly, the quantization of $\mathcal{H}(\mathbf{x})$ is no longer appropriate in this limiting case. This issue is discussed in greater depth in [18].

8.3 The Relation Between Depth and Surface Luminance

Finally, we use our theory to address a classical issue in shading from the artistic literature. As Leonardo da Vinci observed, “among bodies equal in size and distance, that which shines the more brightly seems to the eye nearer” [19]. Notice, for example, how in the rendered images of Figure 2, our own visual systems are not able to precisely infer the depth map of the drapery surface, although there is a strong sense that the bright regions are hills and the dark regions are valleys. Artists often take advantage of this “darker is deeper” heuristic when rendering drapery surfaces [20].

The Shape from Shading algorithm developed in this paper also (at first glance) suggests a heuristic that darker surface nodes tend to be deeper. To examine the validity of this heuristic, we again consider the smooth depth map and rendered images of Figure 2. Figure 8 shows a scatter plot of luminance-depth pairs for these images. The plots suggest that, on average, darker surface elements do have greater depth. However, the large variance of this relation implies that the heuristic is unreliable.

A few examples should clarify why the heuristic can fail. Wrinkles on a person’s face or cracks in the floor are dark but not deep. Here, darkness results from small surface aperture, rather than large depth. Conversely, a surface such as a large valley could have great depth but it need not be dark. In particular, if the width of a valley were much greater than the depth, then the luminance of the valley could be nearly as great as that of the surrounding hills. The conclusion is that darkness is neither necessary nor sufficient for deepness, but rather, is coupled to the surface aperture function.

9 Discussion

There are many types of Shape from Shading problems. The most basic is to distinguish the sky from the ground. We have discussed how a Dominating Sky Principle, which holds that the sky is much brighter than the ground in natural environments, could allow a flying organism to orient itself. A simple visual system would be sufficient for such a task.

A slightly more complex visual system could distinguish different brightnesses coming from the surfaces in a scene. An organism might behave differentially toward these different brightnesses. In particular, if an organism

tended to move toward relatively low (or high) brightnesses, then it would on average move into (or out of) holes or shadows. Whether or not this behavior was advantageous to the organism would depend, of course, on the particular organism and its environment.

A more complex visual system would be able to use the variations in brightness to compute geometric structure of surfaces in a scene. There are many different lighting conditions under which an organism might solve this problem. The classical example of a point light source has been studied in great detail. Another example is a scene illuminated by nearby light source, such as a lamp or fire. For such a scene, a model relating surface luminance and scene geometry would have to take account of how the lighting varies with the position of a surface element in the scene.

In this paper we have considered scenes which are illuminated by a uniform diffuse light source, such as the sky on a cloudy day. We have developed, from first principles, a method for computing a depth map of a Lambertian surface from an image. The first half of the paper is an argument that surface luminance depends primarily on the surface aperture function and that the surface normal is a secondary factor.

In the second half of the paper, we discuss local visibility constraints on the geometry of ambient light. An algorithm for computing a depth map is derived from these constraints. The algorithm uses simple operations at each computational node, and local connectivity between nodes. Conceptually, the geometric constraints which allow one to compute Shape from Shading are along the surface for the classical point source situation, but above the surface for a diffuse light source. It is nevertheless pleasing that this latter, seemingly more complex situation, still admits efficient solutions.

10 Appendix A

We begin by addressing a folk belief held by many, that, by a fortuitous arrangement of surfaces, it could happen a surface element is brighter than the uniform, hemispheric source. The following lemma shows this belief to be unfounded. From the lemma, we derive the Dominating Sky Principle.

Lemma 3 $\forall \mathbf{x} \in \partial\mathcal{F}, \quad I_{out}(\mathbf{x}) \leq I_{\mathcal{D}}$

Proof : Let \mathbf{x}^* be the brightest surface point in the scene, so that $I_{out}(\mathbf{x}) \leq I_{out}(\mathbf{x}^*)$ for any $\mathbf{x} \in \partial\mathcal{F}$. Suppose $I_{\mathcal{D}} < I_{out}(\mathbf{x}^*)$ (we will derive a contradic-

tion). From the Radiosity Equation,

$$\begin{aligned} I_{out}(\mathbf{x}^*) &= \frac{\rho}{\pi} \int_{\mathcal{V}(\mathbf{x}^*)} I_{\mathcal{D}} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}^*) d\Omega + \frac{\rho}{\pi} \int_{\mathcal{V}_C(\mathbf{x}^*)} I_{out}(\mathbf{x}^*, \mathbf{L}) \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) d\Omega \\ &< \frac{\rho}{\pi} \int_{\mathcal{H}(\mathbf{x}^*)} I_{out}(\mathbf{x}^*) \mathbf{L} \cdot \mathbf{N}(\mathbf{x}^*) d\Omega \end{aligned}$$

Thus, $I_{out}(\mathbf{x}^*) < \rho I_{out}(\mathbf{x}^*)$, which is a contradiction since $0 \leq \rho \leq 1$. \square

From the Radiosity Equation and the previous lemma, we have

$$\begin{aligned} I_{out}(\mathbf{x}) &= \frac{\rho}{\pi} \int_{\mathcal{V}(\mathbf{x})} I_{\mathcal{D}} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) d\Omega + \frac{\rho}{\pi} \int_{\mathcal{V}_C(\mathbf{x})} I_{out}(\Pi(\mathbf{x}, \mathbf{L})) \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) d\Omega \\ &\leq \frac{\rho}{\pi} \int_{\mathcal{H}(\mathbf{x})} I_{\mathcal{D}} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) d\Omega \\ &= \rho I_{\mathcal{D}}, \text{ which proves the Dominating Sky Principle.} \end{aligned}$$

It is important to observe that the bounds of the Dominating Sky Principle are attained only under exceptional circumstances. For any $\mathbf{x} \in \mathcal{F}$, the upper bound would be attained only if all surfaces visible from \mathbf{x} had luminance equal to $I_{\mathcal{D}}$. However, if $\rho < 1$, then (by the Dominating Sky Principle) this last condition would be met only in the trivial case that no surfaces were visible from \mathbf{x} .

The lower bound of the Dominating Sky Principle would be attained in the limiting case that none of the diffuse source were visible from \mathbf{x} and those surfaces that were visible from \mathbf{x} all had zero luminance.

11 Appendix B

(Proof of Lemma 2): Given $\mathbf{N}(\mathbf{x})$ and $A(\mathbf{x})$, define a spherical coordinate parameterization of $\mathcal{H}(\mathbf{x})$ with a pole at $\mathbf{N}(\mathbf{x})$ as follows.

$$\mathbf{L}(\theta, \phi) : [0, 2\pi] \times [0, \frac{\pi}{2}] \rightarrow \text{unit sphere}$$

where

$$\begin{aligned} \mathbf{L}(\theta, \phi) \cdot \mathbf{N}(\mathbf{x}) &= \cos\phi \\ d\Omega &= \sin\phi d\phi d\theta \end{aligned}$$

Let ϕ_{min} and ϕ_{max} be the unique real numbers such that

$$\begin{aligned} A(\mathbf{x}) &= \int_0^{2\pi} \int_0^{\phi_{max}} \sin\phi \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_{\phi_{min}}^{\frac{\pi}{2}} \sin\phi \, d\phi \, d\theta \quad . \end{aligned}$$

A straightforward calculation shows that

$$A(\mathbf{x}) = 1 - \cos\phi_{max} = \cos\phi_{min} \quad . \quad (11)$$

Define

$$\begin{aligned} \mathcal{V}_{max} &\equiv \{\mathbf{L}(\theta, \phi) : 0 \leq \theta \leq 2\pi, 0 < \phi < \phi_{max}\} \\ \mathcal{V}_{min} &\equiv \{\mathbf{L}(\theta, \phi) : 0 \leq \theta \leq 2\pi, \phi_{min} < \phi < \frac{\pi}{2}\} \end{aligned}$$

These sets of directions are the result of taking any given solid angle of directions and first, crowding them around $\mathbf{N}(\mathbf{x})$ to form \mathcal{V}_{max} , and second, crowding them into an annulus around the horizon to form \mathcal{V}_{min} . These definitions imply the following:

$$\frac{1}{\pi} \int_{\mathcal{V}_{min}} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) \, d\Omega \leq \frac{1}{\pi} \int_{\mathcal{V}(\mathbf{x})} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) \, d\Omega \leq \frac{1}{\pi} \int_{\mathcal{V}_{max}} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) \, d\Omega \quad (12)$$

This inequality does not make any topological assumptions about the connectedness or number of holes of $\mathcal{V}(\mathbf{x})$, beyond requiring that the integrals be defined.

A straightforward calculation shows

$$\begin{aligned} \frac{1}{\pi} \int_{\mathcal{V}_{min}} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) \, d\Omega &= \cos^2\phi_{min} \\ \frac{1}{\pi} \int_{\mathcal{V}_{max}} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) \, d\Omega &= 1 - \cos^2\phi_{max} \end{aligned}$$

Using Equation (11), we substitute the above into Equation (12) yielding

$$A(\mathbf{x})^2 \leq \frac{1}{\pi} \int_{\mathcal{V}(\mathbf{x})} \mathbf{L} \cdot \mathbf{N}(\mathbf{x}) \, d\Omega \leq A(\mathbf{x})(2 - A(\mathbf{x})) \quad . \quad \square$$

Observe that \mathcal{V}_{max} and \mathcal{V}_{min} are quite symmetric. Since such symmetry rarely arises in natural scenes, it follows that the bounds of Lemma 2 are rarely attained.

12 Appendix C

ALGORITHM : given a depth map z , compute the surface aperture, A^* .

$\forall (x, y) \in \mathcal{P}, \quad \forall n \leq 0, \quad \mathcal{V}_n^*(x, y) := \mathcal{H}^*$;

$n := 0$;

REPEAT

$\forall (x, y) \in \mathcal{P},$

 IF $z(x, y) = n$

 Compute $\mathcal{V}_n^*(x, y)$ using the Local Visibility Constraints;

$n := n + 1$;

UNTIL $\forall (x, y) \in \mathcal{P}, \quad n > z(x, y)$;

$\forall (x, y) \in \mathcal{P}, \quad$ Compute $A^*(x, y)$ using Equation (6).

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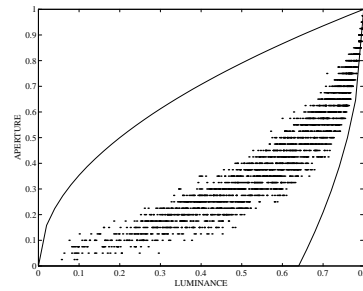
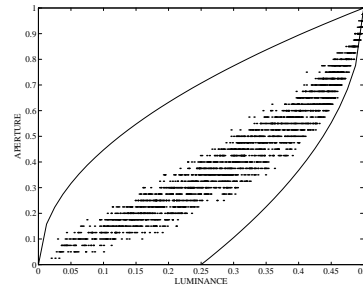
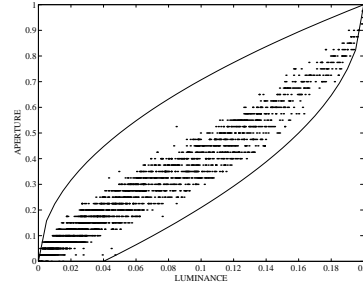
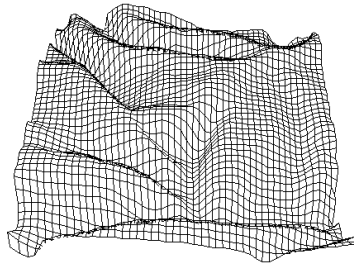


Figure 2: (a) A 50×50 smooth surface is shown. The depth values of the surface span a range of 0 to 25. (b-d) The surface was rendered according to the Radiosity Equation for three values of the albedo (0.2, 0.5, 0.8). The uniform hemispheric source was approximated using 40 point sources. (e-g) A scatter plot of aperture versus luminance is shown for each of the three albedos. The two solid curves represent the upper and lower bounds of the Aperture-Luminance Inequality. Notice that these bounds are quite weak relative to the values found for a typical surface.

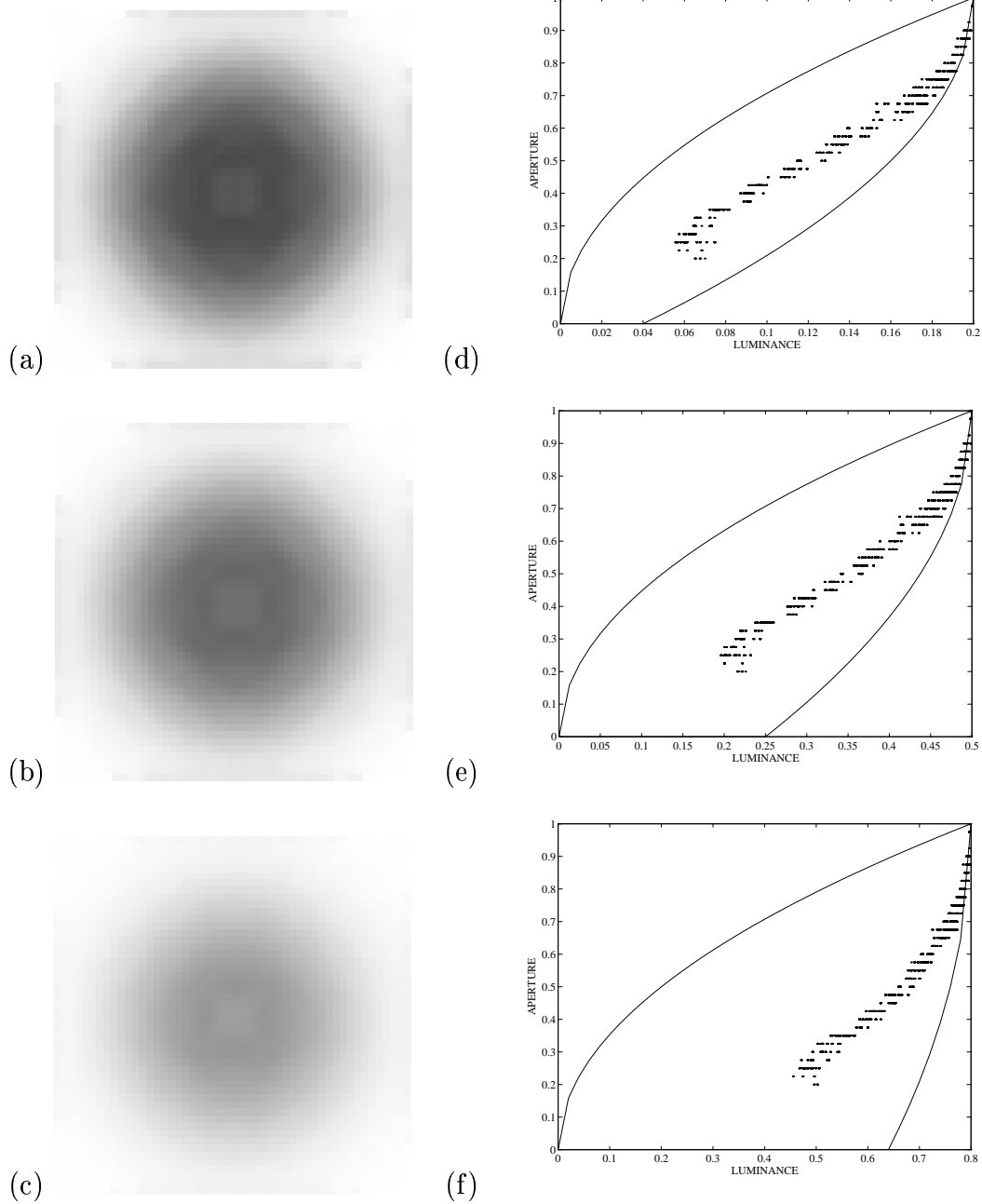


Figure 3: **(a-c)** A concave 50 X 50 Gaussian function having standard deviation 15 pixels and maximum depth 50 was rendered according to the Radiosity Equation using three values of ρ (0.2, 0.5, 0.8). **(d-f)** Scatter plots of aperture versus luminance are shown. (The “leakage” of the data beyond the bounds is a quantization effect.) Clearly, the average aperture for a given luminance varies with ρ . However, the variation of the aperture for a given luminance is roughly the same for all three values of ρ . This suggests that the variation in $A(\mathbf{x})$ for a given value of $I(\mathbf{x})$ is independent of ρ .

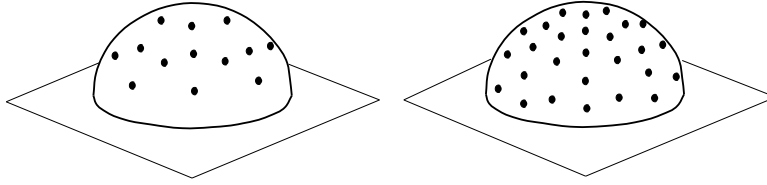


Figure 4: Two different quantizations of \mathcal{H}^* are shown, having 32 and 64 light source directions, respectively. Each of these unit vectors points from the origin to a node in \mathcal{Z}^3 .

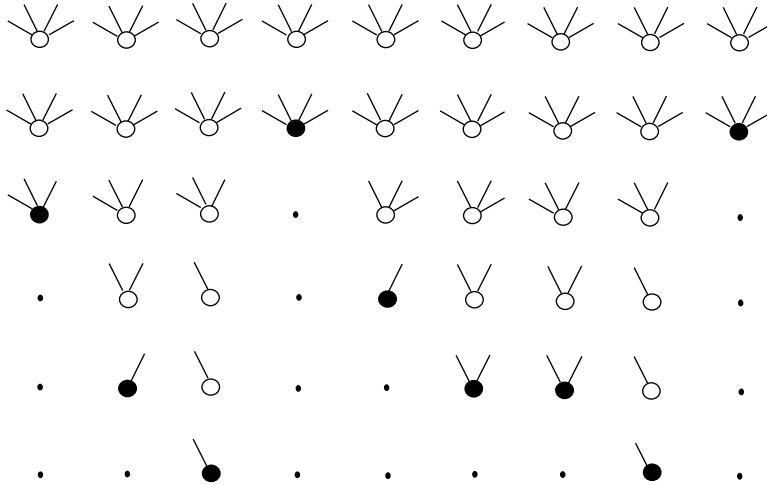


Figure 5: An example of a quantized visibility field is shown for 5 light source directions. Free nodes, \mathcal{F}^* are represented by white discs. Surface nodes, $\partial\mathcal{F}$ are represented by black discs. Nodes below the surface are represented as points. As we will soon see, the local constraints on this field can be used to compute Shape from Shading.

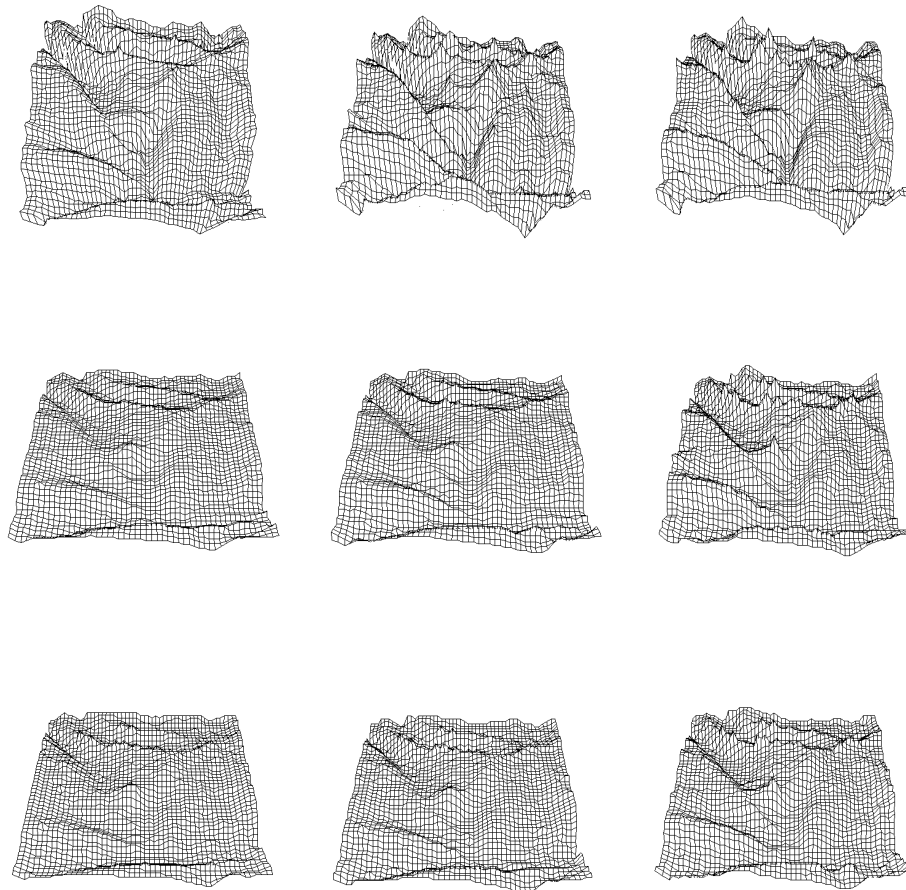


Figure 6: The Shape from Shading on a Cloudy Day algorithm was applied to each of the three rendered images of Fig. 4. Each column above corresponds to a single rendered image of Fig. 4. Each row above corresponds to a different chosen value of ρ (0.2, 0.5, 0.8). The three surfaces along the diagonal thus correspond to the correctly chosen values of ρ . See the text for a discussion of the errors.

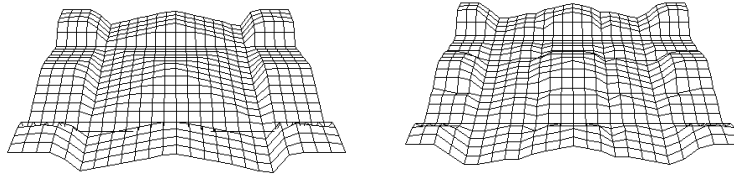


Figure 7: Because z and \mathcal{H}^* are quantized, the two algorithms - aperture from depth (see Appendix C) and depth from aperture - are not inverses of one another. When these two algorithms are run in succession on a original constant depth map, the resulting computed depth is not constant. **(a)** A mesh plot of a computed depth map is shown for an original 50×50 depth map having constant depth ($z \equiv 40$), and using relatively coarse quantization parameters ($M = 32$, and z quantized to unit steps). **(b)** A mesh plot for a finer quantization is shown ($M = 64$, and z quantized to half unit steps).

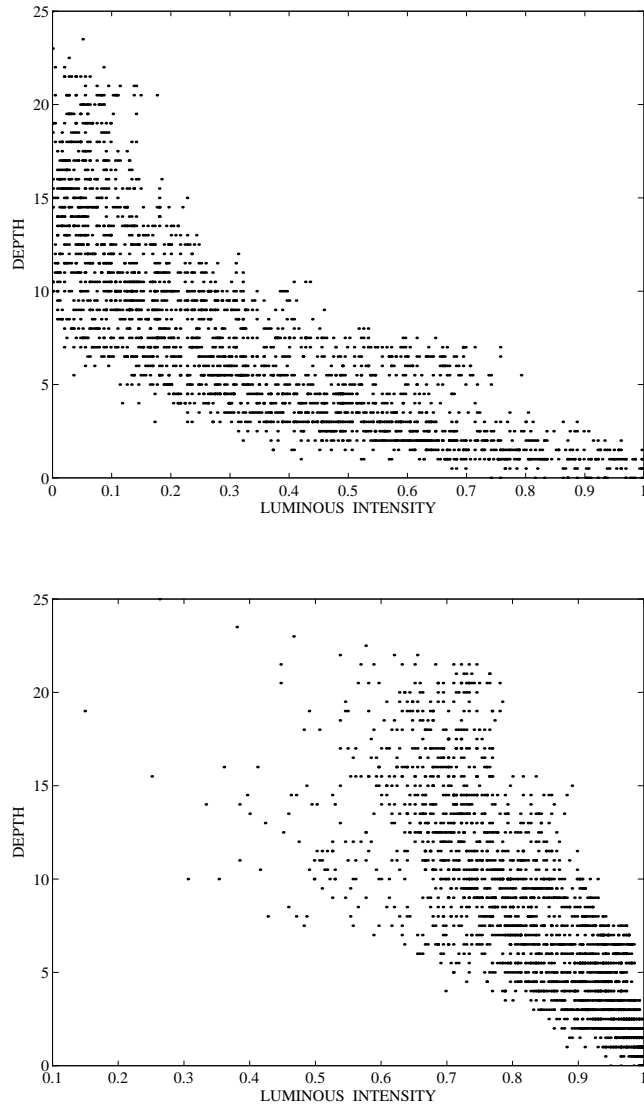


Figure 8: A scatter plot of luminance-depth pairs is shown for the rendered images of Fig. 4. (a) corresponds to $\rho = 0.2$ and (b) corresponds to $\rho = 0.8$. A “darker is deeper” heuristic is only valid in a loose statistical sense. The large variance of the relation implies that the heuristic is unreliable for recovering a depth map from an image.