A Neural Network Wave Formalism

Willard L. Miranker[1]
Yale University
Department of Computer Science

Abstract: Using his path integral methodology, Feynman derived the constructs of quantum mechanics from the Lagrangian form of Newtonian mechanics. By employing a novel Lagrangian form of the canonical neural network equations, we derive analogously a complete wave formalism for the information transmission in neural networks.

Keywords: Lagrangian, least action, neural net, path integral, Schrödinger equation, uncertainty principle, wave equation, wave function, wave-particle duality

1. INTRODUCTION

There are a number of models that describe the flow of information in neural networks, models at various scales of phenomenology (from sub-neuron to neural assembly), Haykin, 1999, Hoppensteadt, 1986, McKenna, Davis, and Zornetzer 1992, Valient, 1994… Most such models, involving as they do, input-output equations, circuit equations, etc. are expressed in terms of the methods of traditional mathematics and physics, and so, for the purposes of analogy, could be viewed as counterparts to classical mechanics. Non-classical methods, such as quantum effects, are also used for such modeling (Hameroff and Penrose, 1996, Penrose, 1997, and Miranker, 2002…). Many believe that the setting and scale are wrong for the relevance of such effects; that the biology just does not support them.

While classical and quantum mechanics have differing scales of relevance and different analytic modeling developments, Feynman showed that the formalism of quantum mechanics is derivable from the Lagrangian form of classical mechanics through use of a path integral methodology that he invented (Feynman, Hibbs, 1965). We show that this derivation can be extended to neural networks. Starting with the Hopfield model of neural networks, we show that a path integral methodology can be used to derive a wave function and a Schrödinger equation for neural networks. While our development might inform the classical/quantum neural network modeling debate (a debate that we hope to avoid), our focus is the mathematical connection between those

[1] Miranker@cs.yale.edu

two frameworks. Note that the methods used here are applicable to dissipative dynamical systems other than those arising out of neural networks.

Key to Feynman's result is the Lagrangian formulation of mechanics with its notions of kinetic and potential energy as well as the principle of least action. These features of mechanics have been shown to exist for the canonical neural network model (Mjolsness, Miranker, 1998), and we begin with a brief review of those ideas. Feynman showed that a stationary phase perturbation analysis of the path integral (as $\hbar \to 0$) delivers the Newtonian path` as the location of the stationary phase. (See also Keller, 1953, Miranker, 1957) We show that an analogous perturbation result prevails for neural networks.

The derivation of a Schrödinger equation for the newly derived wave function of neural transmission motivates introduction of a virtual particle (corresponding to a neuron) of an appropriately specified mass in duality to the wave formalism. De Broglie relations are then introduced and used to suggest a way to determine the scales at work in our development. Finally we derive an uncertainty principle for neural transmission.

2. A LAGRANGIAN THEORY OF NEURAL NETWORKS

2.1 Neuronal input/output

Let $w = (w_1, ..., w_n)$ be the vector of the *n* synaptic weights of a model neuron, and let $v^a = (v_1^a, ..., v_n^a)$ be the vector of inputs. The total neuronal input u is the weighted sum

$$u = \sum_{k=1}^{n} w_k v_k^a , \tag{2.1}$$

and the neuronal output, v^e is a gain function, g (with threshold) of the total input:

$$v^e = g(u). \tag{2.2}$$

2.2 A Lagrangian formulation of neural net dynamics

Since neural net dynamics are dissipative, a so-called greedy variation is used (Mjolsness and Miranker, 1998) to generate a Lagrangian formulation of neural net dynamics including a principle of least action.

In correspondence with the use of the variable *v* to represent the neural output, we use v to express the position variable and $\dot{v}$ the velocity for the Lagrangian

$$L(\dot{v}, v) = K(\dot{v}) - P(v). \tag{2.3}$$

Here $K = K(\dot{v})$ is the kinetic energy and $P = P(v)$ is the potential energy. (Examples of *K* and *P* are given presently.) The action *S* is given by

$$S = \int_{-\infty}^{\infty} L\,dt. \tag{2.4}$$

We extremize S as follows.

(2.5) $$\frac{\delta_G S}{\delta_G v} = \frac{\delta}{\delta v}\int_{-\infty}^{t} L(\dot{v}, v)\,dt .$$

$\delta_G / \delta_G v$ is called a greedy variation since it specifies extremization, not merely for the trajectory as a whole, but for the trajectory at every instant of time. Then we can show (see the Appendix) that

(2.6) $$\frac{\delta_G S}{\delta_G v} = \frac{\partial L(\dot{v}, v)}{\partial \dot{v}} .$$

This should be contrasted with the conventional variation (not greedy) of the action, namely

(2.7) $$\frac{\delta S}{\delta v} = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}}\right) - \frac{\partial L}{\partial t} .$$

Suppose the Lagrangian has the form (a form of relevance to neural net applications)

(2.8) $$L = K(\dot{v}) + \frac{dE(v)}{dt} = K(\dot{v}) + \frac{\partial E(v)}{\partial v}\frac{dv}{dt} .$$

Then using (2.6), the greedy variation of the corresponding action is

(2.9) $$\frac{\partial L}{\partial \dot{v}} = \frac{\partial K}{\partial \dot{v}} + \frac{\partial E}{\partial v} .$$

In equilibrium (where the velocity $\dot{v}$ vanishes), we have $\partial K(\dot{v} = 0) / \partial \dot{v} = 0$. Then extremization of the action *S* yields extremal points of *E*. That is,

(2.10) $$\frac{\delta_G S}{\delta_G v} = \frac{\partial L}{\partial \dot{v}} = 0 \quad \Rightarrow \quad \frac{\partial E}{\partial v} = 0 .$$

2.3 Application to neural nets, the neural net principle of least action

The relaxation neural net is associated with an energy (a Lyapunov) function $E(v)$. For example, take

(2.11) $$E = E(v) = -\frac{1}{2}\sum_{ij} T_{ij} v_i v_j - \sum_i f_i v_i + \sum_i \Phi_i(v_i)$$

(see Hertz, Krogh and Palmer, 1991, Sect. 3.3), and take

(2.12) $$K = \frac{1}{2}\sum_i \dot{u}_i^2 g'(u_i) ,$$

where

(2.13) $$v_i = g(u_i) .$$

In (2.11) T_{ij} corresponds to the synaptic weight between neuron *i* and neuron *j*, f_i is the exogenous input to the network at neuron *i*, and $\Phi' = g^{-1}$ (see (2.2)). With the choices (2.11)-(2.13), extremization of the action *S* via greedy variation (using (2.10)) yields

(2.14) $$\dot{u}_i = -\frac{\partial E}{\partial v_i} = -\sum_j T_{ij} v_j - f_i + \Phi_i'(v_i).$$

(2.14) is precisely the customary neural net dynamics. So (2.3), (2.4), (2.12) and (2.14) are the (dissipative) neural net analogs of the Lagrangian, the action, the kinetic energy and the equations of motion in classical mechanics, respectively. The form of the potential energy is obtained from the last term in (2.8). From (2.10) we see that the *neural net principle of (greedy) least action* is

(2.15) $$\frac{\delta_G S}{\delta_G v} = 0.$$

3. THE WAVE FUNCTION

3.1 The neural net as a path generator

We shall use a more representative label x_k $(x_k = 0,1\ldots)$ for indexing neurons in the *k*-th layer, $k = 0,1,\ldots,N-1$. For clarity, take the gain *g* to be linear and homogeneous. Then from the I/O relations (2.1), (2.2), the output of a neuron, as it depends on propagation of signals from $N-1$ preceding neuronal layers, is expressible as follows.

(3.1) $$g\sum_{x_0}\cdots g\sum_{x_{N-1}}\prod_{k=1}^{N-1} w_{x_k x_{k-1}} v^e_{x_1 x_0}.$$

$v^e_{x_1 x_0}$ is the output of neuron x_0 in layer zero transmitted to neuron x_1 in layer one. For clarity this term is dropped in (3.1). The expression $g^N \prod_{k=1}^{N-1} w_{x_k x_{k-1}}$ from (3.1) represents a succession of outputs strung along a *path through N neuronal layers*, the *k*-th vertex of the path being at neuron x_k in layer *k*.

3.2 Modeling the neuron as a frequency encoder

(2.1), (2.2) are a simplification of actual neuronal information processing that is in fact frequency encoded (see Curtis, Barnes, 1989, Fig. 41-12, p. 589). Namely, what is called the neuronal activity models the *frequency of the actual output*, *a spikey waveform*. So we formally replace the neuronal output in the model by $\exp i(v_k t + 2\pi x_k/\lambda_k)$. Here v_k is the output frequency of the action potential of the *k*-th neuron, t is the time, x_k is distance along its axon (and associated relevant branching processes) and λ_k is the signal wavelength. For clarity we shall drop the term $2\pi x_k/\lambda_k$. Then the *k*-th neuron's output is written simply as

(3.2) $$\exp(iv_k t).$$

3.3 The path integral

Using the frequency encoding of Section 3.2, the product in (3.1) becomes the exponential sum $\exp(i\sum_{k=1}^{N-1} w_{x_k x_{k-1}})$. Now introducing two time scales, $\Delta x / A$ (representing the time to execute the gain function) and $(t_k - t_{k-1})/h$ (representing the time to convey the signal between neuronal layers), where *A* and *h* are appropriate scaling factors, (3.1) becomes

$$\sum_{x_0} g\frac{\Delta x}{A}\cdots\sum_{x_{N-1}} g\frac{\Delta x}{A}\exp\left[i\sum_{k=1}^{N-1} w_{x_k x_{k-1}}(t_k - t_{k-1})/h\right]. \tag{3.3}$$

The value of *A* is specified in (4.6), but the value of *h* (that in quantum mechanics is the Planck constant $\hbar$) is unknown. Determining a value for *h* is discussed in Section 5.2.

For clarity, replace $w_{x_k x_{k-1}}$ in (3.3) with $w(x_k - x_{k-1})$, a function of one variable. Since we shall be taking the limit as the neural network becomes dense, $w(x_k - x_{k-1})$ will be replaced by $w(\xi)$, a spatially continuous version of the synaptic weights. Then (3.3) has the form of a collection of Riemann sums corresponding to the following collection of integrals.

$$\int \overset{\infty}{\underset{-\infty}{\cdots}} \int e^{\frac{i}{h}S} g\frac{dx_1}{A}\cdots g\frac{dx_{N-1}}{A} \tag{3.4}$$

and

$$S = \int_0^t w(\xi)d\xi. \tag{3.5}$$

These relations define a path integral as $N \to \infty$ (Feynman, Hibbs, 1965, Sect. 2-4). In (3.4), (3.5), *t* is a fixed value of time. The relation $t = N\Delta t$, where $t_k - t_{k-1} = h\Delta t, k = 1,...,N$ connects the t_k in (3.3) with *t* in (3.5). So taking the limit in (3.3) as $N \to \infty$ formally defines a path integral (with respect to an appropriate functional measure μ) that expresses propagation in (a continuum) neural net, namely

$$\int_{paths} e^{\frac{i}{h}S} \mu(dx). \tag{3.6}$$

3.4 The kernel $\Gamma(b;a)$

Let $a = (v_a, t_a)$ and $b = (v_b, t_b)$ denote points between which a path (as defined in Section 3.1) of the neural net dynamics passes (say from *a* to *b* as time increases). Then for *h* and $A = A(\varepsilon)$ as suitable constants, define the kernel $\Gamma(b;a)$ by means of a path integral as follows.

$$\Gamma(b;a) = \lim_{\varepsilon\to 0}\frac{1}{A}\int \overset{\infty}{\underset{-\infty}{\cdots}} \int e^{\frac{i}{h}S[b,a]}\frac{dv_1}{A}\cdots\frac{dv_{N-1}}{A}. \tag{3.7}$$

Here $v_1,\ldots,v_{N-1}$ is a uniform partition of (v_a,v_b) with mesh width $\varepsilon = (v_a - v_b)/N$, and

(3.8) $$S[b,a] = \int_{t_a}^{t_b} L(\dot{v},v)dt.$$

To interpret (3.7), (3.8), replace the right hand side of (3.8) by a Riemann sum with respect to a partition of (t_a,t_b) also with mesh width ε, in particular by

(3.9) $$\sum_{\mathrm{j=0}}^{\mathrm{N\text{-}1}} L\left(\frac{v_{j+1}-v_j}{\varepsilon},\frac{v_{j+1}+v_j}{2}\right)\varepsilon.$$

Then the integral dv_i within (3.7) becomes

(3.10) $$\int_{-\infty}^{\infty}\exp\frac{i}{h}\left[L\left(\frac{v_{i+1}-v_i}{\varepsilon},\frac{v_{i+1}+v_i}{2}\right)+L\left(\frac{v_i-v_{i-1}}{\varepsilon},\frac{v_i+v_{i-1}}{2}\right)\right]\frac{dv_i}{A}.$$

Recall that in the case of the neural net dynamics at hand, $L = K + \dfrac{dE}{dt}$, $E = E(v)$, and $K = \dfrac{1}{2}\sum_i \dot{u}_i^2 g'(u_i)$ with $v_i = g(u_i)$. Now suppose that the limit in (3.7) exists. Then it defines a path integral (with an appropriate functional measure μ) denoted by

(3.11) $$\Gamma(b;a) = \int_a^b e^{\frac{i}{h}S[b,a]}\mu(dv).$$

3.5 The wave function

Let $\psi(v,t)$ denote the wave function of the neural net. It is defined by the condition that it have the following property of evolution in time.

(3.12) $$\psi(v_2,t_2) = \int_{-\infty}^{\infty}\Gamma(v_2,t_2;v_1,t_1)\psi(v_1,t_1)dv_1,\quad t_2 > t_1,$$

where the kernel Γ is given in (3.7). $\psi(v,0)$ is prescribed as an arbitrary probability amplitude, namely,

(3.13) $$\int_{-\infty}^{\infty}|\psi(v,0)|^2 dv = 1.$$

This normalization property will be maintained for all $t > 0$ (conservation of probability). A proof follows as in the case of mechanics, since up to a phase, $\psi(v,t)$ is seen to satisfy the Schrödinger equation (see (4.10) and (4.11)).

3.6 The classical limit

In the case of mechanics, the Lagrangian has the form $L = m\dot{x}^2/2 - V(x,t)$. In this case, *h*, which appears in the exponential in (3.11), is the Planck constant $\hbar$, and the corresponding path integral is used to define the Schrödinger wave function. It is shown by means of a stationary phase argument (Feynman, Hibbs (1965), Sect. 2-3.) that in the limit as $\hbar \to 0$, the contributions to the value of the path integral cancel except where the action S is stationary (i.e., where $\delta S/\delta x = 0$). To implement the stationary phase calculation, a variation in the action is performed, and using the result in (2.7), the Lagrangian form of the equations of classical mechanics emerge. That is, classical mechanics emerges from wave mechanics in the limit as $\hbar \to 0$.

The same argument can be applied to the path integral (3.11), except that we make the action stationary (i.e., we conduct the stationary phase calculation, corresponding to $h \to 0$) by performing a greedy variation (where $\frac{\delta_G S}{\delta_G v} = \frac{\partial L(\dot{v},v)}{\partial \dot{v}} = 0$ (cf. (2.6), (2.15)). Using L as given in (2.8) and in the neural net case (2.11)-(2.13), we recover the neural net dynamics (2.14) by this limiting process. So the (customary) equations of neural net dynamics emerge from the path integral wave description of the neural net in the limit as $h \to 0$, provided that the action is appropriately defined and extremized (in the greedy sense) as in Section 2.

4. A NEURAL NET SCHRöDINGER EQUATION

A neural net Schrödinger equation that describes the time evolution of the neural net wave function $\psi(v,t)$ is derivable, moreover along the lines in Feynman, Hibbs (1965). We give a summary of the results.

Let $\Gamma(i+1;i)$ denote the kernel of (3.7) corresponding to the passage of information in an infinitesimal time interval ε between two locations (two v values) indexed by i+1 and i, respectively. Then consider the following approximation for $\Gamma(i+1;i)$.

(4.1)
$$\Gamma(i+1;i) = \frac{1}{A}\exp\left[\frac{i}{h}\varepsilon L\left(\frac{v_{i+1}-v_i}{\varepsilon},\frac{v_{i+1}+v_i}{2}\right)\right].$$

Using (3.12) and (4.1), take the following approximation for $\psi(v,t+\varepsilon)$.

(4.2)
$$\psi(v,t+\varepsilon) = \frac{1}{A}\int_{-\infty}^{\infty}\exp\left[\frac{i}{h}\varepsilon L\left(\frac{v-v_1}{\varepsilon},\frac{v+v_1}{\varepsilon}\right)\right]\psi(v_1,t)\,dv_1.$$

The Lagrangian (cf. (2.8)) is analogously approximated:

(4.3)
$$L\left(\frac{v-v_1}{\varepsilon},\frac{v+v_1}{2}\right) = \frac{1}{2g'(u)}\left(\frac{v-v_1}{\varepsilon}\right)^2 + E_v\left(\frac{v+v_1}{2}\right)\frac{v-v_1}{\varepsilon}.$$

Note that $g'(u) = g'(g^{-1}(v)) \approx g'(g^{-1}(\frac{v+v_1}{2}))$. Then inserting (4.3) into (4.2) gives

(4.4)
$$\psi(v,t+\varepsilon) = \int_{-\infty}^{\infty}\frac{1}{A}\left\{\exp\left[\frac{i}{2hg'(u)}\frac{(v-v_1)^2}{\varepsilon}\right]\right\}\left\{\exp\left[\frac{i}{h}(v-v_1)E_v\left(\frac{v+v_1}{2}\right)\right]\right\}\psi(v_1,t)\,dv_1.$$

Setting $v_1 = v + \eta$, (4.4) becomes

$$(4.5) \qquad \psi(v, t+\varepsilon) = \int_{-\infty}^{\infty} \frac{1}{A} \exp\left[\frac{i}{2hg'(u)} \frac{\eta^2}{\varepsilon} - \frac{i}{h} \eta E_v \left(\frac{2v+\eta}{2} \right) \right] \psi(v+\eta, t) d\eta .$$

By expanding (4.5) in Taylor series in ε, and then equating terms of equal order in ε, we find the following results. (Details are given in Miranker, 2002.) The zero-th order equation determines *A*. Namely

$$(4.6) \qquad A = \left(\frac{2\pi i h \varepsilon}{m(v)} \right)^{\frac{1}{2}},$$

where $m(v)$ is an appropriate mass. In particular,

$$(4.7) \qquad m = m(v) = 1/g'(g^{-1}(v)).$$

The first order equation in ε is vacuous. The second order equation in ε yields the wave equation sought (a neural net Schrödinger equation). Namely

$$(4.8) \qquad \frac{h}{i} \frac{\partial \psi}{\partial t} = \frac{h^2}{2m(v)} \frac{\partial^2 \psi}{\partial v^2} + \frac{h}{i} E_v \frac{\partial \psi}{\partial v} - V\psi .$$

Here

$$(4.9) \qquad V = V(v) = \frac{E_v^2(v)}{2m(v)} + \frac{ih}{2m(v)} E_{vv}(v) .$$

Setting

$$(4.10) \qquad \varphi = \psi \exp\left[\frac{h}{4im(v)} \int^{v} \frac{E_v(v')}{m(v')} dv' \right],$$

a calculation shows that φ satisfies the Schrödinger equation of quantum mechanics. In particular,

$$(4.11) \qquad \frac{\partial \varphi}{\partial t} = -\frac{h}{2im(v)} \frac{\partial^2 \varphi}{\partial v^2} + U(v)\varphi ,$$

where *U(v)* is the following complex valued potential.

$$(4.12) \qquad U(v) = \frac{1}{m}\left(E_v^2 + \frac{1}{2} E_{vv} \right) + \frac{i}{2m}\left(\frac{hm_v}{m^2} E_v - \frac{1-4m}{h} E_v^2 - hE_{vv} \right).$$

(For an example of a complex valued quantum potential, see Kleinert, 1995, Sect. 14.5.)

5. AN UNCERTAINTY PRINCIPLE

We give two derivations of an uncertainty principle associated with our neural net wave function. The first follows from the Heisenberg inequality and the quantum mechanical measurement hypothesis, and the second from de Broglie relations.

5.1 From the quantum mechanical measurement and the Heisenberg inequality

The Heisenberg inequality is a universal result concerning L^2-functions and their Fourier transforms. This inequality delivers the uncertainty principle directly from the wave function in its role of probability amplitude when the measurement of a variable in quantum mechanics is taken to be the expected value of the variable's corresponding operator. See Dym, McKean, 1972 for a succinct presentation of these ideas. So having derived a neural net wave function, we can apply the same argument to derive an uncertainty principle associated with our neural net wave function.

5.2 From de Broglie relations

De Broglie relations permit the association of a particle in dual correspondence to the wave formalism thus far developed. The mass of this putative particle, $m(v)$, specified in (4.7), is v (wave frequency) dependent. The de Broglie relations are

$$E = \frac{h}{2\pi} v \quad \text{and} \quad p = \frac{h}{\lambda}, \tag{5.1}$$

where recall that v denotes the wave frequency, and p denotes the particle momentum. Using $E = p^2 / 2m$ to combine these relations, and setting $k = 2\pi / \lambda$, we find

$$\frac{h}{2\pi} = \frac{2vm(v)}{k^2}. \tag{5.2}$$

With a notion of a particle and of momentum (from (5.1)) to associate with the position x, we have an uncertainty principle, namely

$$\Delta x \Delta p \approx h. \tag{5.3}$$

Of course the value of h is as yet unknown, but the relation (5.2) suggests how it may be determined. Note that the wave number $k = k(v)$ is frequency dependent. To see how this uncertainty principle comes about, use (5.1) and write x/λ as xp/h. Then an uncertainty in simultaneous measurement of x and p corresponds to an uncertainty in simultaneous measurement of x and λ. To measure x and λ simultaneously, we must measure two values of x (x_1 and x_2 say, where $\lambda = x_2 - x_1$). Fig. 5.1 shows a ruler being used to simultaneously measure the location of two points noted on the x-axis. In the figure, the zero point of the ruler is lined up with x_1. Then x_2 will almost always fall between two ruler tic marks as illustrated. The distance between these marks is the uncertainty in the measurement in question. If we knew the ruler, we would know h.

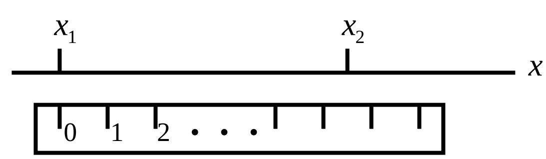

Figure 5.1: The uncertainty principle illustrated

APPENDIX

In this appendix we shall demonstrate (2.6), namely that

(A1) $$\frac{\delta_G S}{\delta_G v} = \frac{\partial L(\dot{v}, v)}{\partial \dot{v}} .$$

\`

Let us take the greedy variation of *S* about a function v^*. Let $v = v^* + \varepsilon u$, where $\varepsilon > 0$ and *u* is an arbitrary function. We have

(A2) $$\begin{aligned}\frac{\delta_G S}{\delta_G v} &= \frac{\delta}{\delta v}\int_{-\infty}^{t} L(\dot{v}^*, v^*)\,d\tau \\ &= \lim_{\varepsilon \to 0}\int_{-\infty}^{t} \frac{L(\dot{v}^* + \varepsilon \dot{u}, v^* + \varepsilon u) - L(\dot{v}^*, v^*)}{\varepsilon}\,d\tau \\ &= \int_{-\infty}^{t}\left(\frac{\partial L(\dot{v}^*, v^*)}{\partial v}u - \frac{\partial L(\dot{v}^*, v^*)}{\partial v}u\right)d\tau \\ &= \frac{\partial L}{\partial \dot{v}}u\Big|_{-\infty}^{t} + \int_{-\infty}^{t}\left(\frac{\partial L}{\partial v} - \frac{\partial}{\partial v}\left(\frac{\partial L}{\partial \dot{v}}\right)\right)d\tau .\end{aligned}$$

(The point of departure of this derivation of greedy variation from the customary one is the replacement in the upper limit of integration of ∞ by *t*.)

Choose $u(\tau)$ to be the function $\mathrm{u}(\tau) = \exp(-(\mathrm{t} - \tau)^2 / a)$, where $a > 0$. We see that

(A3) $$u(t) = 1,\ \lim_{a \to 0} u(\tau) = 0,\ \text{ and } u(-\infty) = 0 .$$

Then (A2) gives

(A4) $$\begin{aligned}\frac{\delta_G S}{\delta_G v} &= \frac{\delta}{\delta v}\int_{-\infty}^{t} L(\dot{v}^*, v^*)\,d\tau \\ &= \frac{\partial L}{\partial \dot{v}}u(t) - \frac{\partial L}{\partial \dot{v}}u(-\infty) + \int_{-\infty}^{t}\left(\frac{\partial L}{\partial v} - \frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial \dot{v}}\right)\right)u\,d\tau .\end{aligned}$$

Taking the limit here as $a \to 0$ and using (A3), we obtain

(A5) $$\frac{\delta_G S}{\delta_G v} = \frac{\delta}{\delta v}\int_{-\infty}^{t} L(\dot{v}^*, v^*)\,d\tau = \frac{\partial L(\dot{v}^*, v^*)}{\partial \dot{v}} ,$$

demonstrating (2.6).

REFERENCES

Curtis, H. and Barnes, N. (1989), *Biology,* (New York: Worth).

Dym, H. and McKean, H.P. (1972), *Fourier Series and Integrals* (New York: Academic Press)

Feynman, R. P. and Hibbs, A. R. (1965), *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill Book Company).

Hammeroff, S. and Penrose, R. (1996), 'Conscious events as orchestrated time-space selections', J. Consciousness Studies, **3**, pp. 36-53.

Haykin, S. (1999), *Neural Networks A Comprehensive Foundation* (Upper Saddle River: Prentice-Hall).

Hertz, J., Krogh, A., and Palmer, R. (1991), *Introduction to the Theory of Neural Computation* (Redwood City: Addison -Wesley).

Hopfield, J. (1982), 'Neural networks and physical systems with emergent collective computational abilities', *Proceedings of the National Academy of Sciences, USA*, **79**, pp. 2554-2558.

Hoppensteadt, F. (1986), *Introduction to the Mathematics of Neurons* (Cambridge: Cambridge University Press).

Keller, J. (1953), 'The geometrical theory of diffraction' *Proc. of the Symp. on Microwave Optics,* **1**, Eaton Electronics Laboratory, McGill University, Montreal.

Kleinert, H., (1995), *Path Integrals in Quantum Mechanics Statistics and Polymer Physics* (Singapore: World Scientific).

McKenna, T, Davis, J. and Zornetzer, S., Eds. (1992), *Single Neuron Computation (*Boston: Academic Press).

Miranker, W.L. (1957) 'Parametric theory of $\Delta u + k^2 u = 0$', *Archive for Rational Mechanics and Analysis*, **1**, 2, 139-153.

Miranker, W.L. (1997), 'Interference effects in computation', SIAM Review, **39**, pp. 630-643.

Miranker, W.L. (2002), 'A quantum theory of consciousness', J. Consciousness Studies, **9**, pp.

Miranker, W.L. (2002), 'Path Integrals of Information', Yale Univ. DCS/TR-1215.

Mjolsness, E. and Miranker, W.L., (1998), 'A Lagrangian formulation of neural networks, Part I, 'Theory and analog dynamics, Part II, Clocked objective functions and applications', Neural, Parallel and Scientific Computations, **6**, pp.

Penrose, R. (1997), *The Large, the Small and the Human Mind* (Cambridge: Cambridge University Press).

Valient, L. (1994), *Circuits of the Mind* (New York: Oxford University Press).