Analogues for Bessel Functions of the Christoffel-Darboux Identity

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Abstract

We derive analogues for Bessel functions of what is known as the Christoffel-Darboux identity for orthonormal polynomials:

$$\sum_{k=1}^{\infty} 2\,(\nu+k)\; J_{\nu+k}(w)\; J_{\nu+k}(z) = \frac{wz}{w-z}\Big(J_{\nu+1}(w)\; J_{\nu}(z) - J_{\nu}(w)\; J_{\nu+1}(z)\Big)$$

and

$$2\,(\nu+1)\sum_{k=1}^{\infty} 2\,(\nu+2k)\; J_{\nu+2k}(w)\; J_{\nu+2k}(z) = \frac{w^2\,z^2}{w^2-z^2}\Big(J_{\nu+2}(w)\; J_{\nu}(z) - J_{\nu}(w)\; J_{\nu+2}(z)\Big)$$

for any distinct nonzero complex variables w and z, and any complex number ν, where J_μ is the Bessel function of the first kind of order μ, for any complex number μ. We also provide certain straightforward consequences of these identities.

1 Introduction

Via some brief manipulations, the present note derives fairly simple expressions for

$$\sum_{k=1}^{\infty} 2\,(\nu+k)\; J_{\nu+k}(w)\; J_{\nu+k}(z) \tag{1}$$

and

$$\sum_{k=1}^{\infty} 2\,(\nu+2k)\; J_{\nu+2k}(w)\; J_{\nu+2k}(z) \tag{2}$$

for any distinct nonzero complex variables w and z, where ν is a complex number, and J_μ is the Bessel function of the first kind of order μ, for any complex number μ (see, for example, [9]).

The simplified expressions for (1) and (2) are analogues for Bessel functions of what is known as the Christoffel-Darboux identity for orthonormal polynomials (see, for example, [5]). See [8] for an application of the results of the present note.

A number of mathematicians have published simplified expressions for the series (1) and (2); the initial publications include [1], [4], [10], and the 1937 edition of [6]. See Section 16.32 of [9] for a description of Kapteyn's and Watson's researches, and Section 11.13 of [6] for a description of Titchmarsh's researches. See also [10] and [11] for a comprehensive account of the literature concerning (1), (2), including Bateman's, Kapteyn's, Titchmarsh's, Watson's, and Wilkins' results, and their relevance in the theory of what are known as Neumann series. See [2] for a treatment of certain related series that is somewhat similar to that of the present note. See [7] for analogues for Bessel functions of the Christoffel-Darboux identity that are based on differential equations; the present note relies on recurrence relations (that is to say, difference equations).

The present note has the following structure: Section 2 summarizes several classical facts about Bessel functions, Section 3 provides proof of a rather trivial fact about Bessel functions, Section 4 derives simplified expressions for (1) and (2) directly from the facts in Sections 2 and 3, and Section 5 generalizes the results of Section 4.

2 Preliminaries

This section provides several well-known facts about Bessel functions. All of these facts can be found, for example, in [9].

For any complex number ν, we define J_ν to be the (generally multiply valued) Bessel function of the first kind of order ν (see, for example, [9]).

The following theorem states a basic symmetry of a Bessel function.

Theorem 1 *Suppose that ν is a complex number.*

Then,

$$J_\nu(z\,e^{\pi i}) = e^{\pi i \nu}\,J_\nu(z) \tag{3}$$

for any nonzero complex variable z.

Proof. Formula 1 of Section 3.62 in [9] provides a slightly more general formulation of (3). □

The following theorem states that two Bessel functions are orthogonal on $(0, \infty)$ with respect to the weight function $w(x) = \frac{1}{x}$ when their orders differ by a nonzero even integer.

Theorem 2 *Suppose that ν is a complex number.*

Then,

$$\int_0^\infty dx\,\frac{1}{x}\,J_{\nu+2j}(x)\,J_{\nu+2k}(x) = \begin{cases} 0, & j \neq k \\ \frac{1}{2\,(\nu+2j)}, & j = k \end{cases} \tag{4}$$

for any integers j and k such that the real part of $\nu + j + k$ is positive.

Proof. Formula 7 of Section 13.41 in [9] provides a slightly more general formulation of (4). □

The following theorem provides what is known as the Poisson integral representation of a Bessel function.

Theorem 3 *Suppose that ν is a complex number such that the real part of ν is greater than $-\frac{1}{2}$.*

Then,

$$J_\nu(z) = \frac{z^\nu}{\sqrt{\pi}\,2^\nu\,\Gamma(\nu+1/2)}\int_{-1}^1 du\,e^{iuz}\,(1-u^2)^{\nu-1/2} \tag{5}$$

for any complex variable z, where Γ is the gamma (factorial) function.

Proof. Formula 4 of Section 3.3 in [9] provides an equivalent formulation of (5). □

The following three theorems provide recurrence relations for Bessel functions and their derivatives.

Theorem 4 *Suppose that ν is a complex number.*

Then,

$$\frac{d}{dz} J_{\nu+1}(z) = J_\nu(z) - \frac{\nu+1}{z}\, J_{\nu+1}(z) \tag{6}$$

for any nonzero complex variable z.

Proof. Formula 3 of Section 3.2 in [9] provides an equivalent formulation of (6). □

Theorem 5 *Suppose that ν is a complex number.*

Then,

$$\frac{d}{dz} J_\nu(z) = \frac{\nu}{z}\, J_\nu(z) - J_{\nu+1}(z) \tag{7}$$

for any nonzero complex variable z.

Proof. Formula 4 of Section 3.2 in [9] provides an equivalent formulation of (7). □

Theorem 6 *Suppose that ν is a complex number.*

Then,

$$\frac{2\nu}{z}\, J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z) \tag{8}$$

for any nonzero complex variable z.

Proof. Formula 1 of Section 3.2 in [9] provides an equivalent formulation of (8). □

The following corollary is an immediate consequence of (8).

Corollary 7 *Suppose that ν is a complex number such that $\nu \neq 1$ and $\nu \neq -1$.*

Then,

$$\frac{2\nu}{z^2}\, J_\nu(z) = \frac{1}{2\,(\nu-1)}\, J_{\nu-2}(z) + \frac{\nu}{\nu^2-1}\, J_\nu(z) + \frac{1}{2\,(\nu+1)}\, J_{\nu+2}(z) \tag{9}$$

for any nonzero complex variable z.

3 A simple technical fact

The purpose of this section is to provide proof of Theorem 9, stating the unsurprising and entirely trivial fact that, for any nonzero complex variable z, $J_{\nu+k}(z)$ tends to 0 as k tends toward ∞ through the integers.

The following theorem bounds the absolute value of the gamma (factorial) function evaluated at a certain complex number.

Theorem 8 *Suppose that ν is a complex number.*

Then,

$$\left|\Gamma\left(\nu + k + \frac{1}{2}\right)\right| \geq (k-n)!\, \left|\Gamma\left(\nu + n + \frac{1}{2}\right)\right| \tag{10}$$

for any integers k and n such that $k > n > |\nu| + \frac{1}{2}$, where Γ is the gamma (factorial) function.

Proof. We have that

$$\left|\Gamma\left(\nu+k+\frac{1}{2}\right)\right| = \left|\Gamma\left(\nu+n+\frac{1}{2}\right)\right| \prod_{j=1}^{k-n}\left|\nu+k-j+\frac{1}{2}\right| \tag{11}$$

for any integers k and n such that $k > n > |\nu| + \frac{1}{2}$. Furthermore,

$$\left|\nu+k-j+\frac{1}{2}\right| \geq k-n-j+1 \tag{12}$$

for any integers j, k, and n such that $k > n > |\nu| + \frac{1}{2}$ and $j \leq k - n$. Combining (11) and (12) yields (10). □

The following technical theorem is the principal purpose of this section.

Theorem 9 *Suppose that ν is a complex number.*

Then, for any nonzero complex variable z, $J_{\nu+k}(z)$ tends to 0 as k tends toward ∞ through the integers.

Proof. We have that

$$\left|1-u^2\right| \leq 1 \tag{13}$$

for any $u \in [-1, 1]$. Using (13), we obtain that

$$\left|(1-u^2)^{\nu-1/2}\right| \leq 1 \tag{14}$$

for any $u \in [-1, 1]$.

Moreover,

$$\left|e^{iuz}\right| \leq e^{|u|\,|\operatorname{Im} z|} \tag{15}$$

for any real number u, and any nonzero complex variable z, where $\operatorname{Im} z$ is the imaginary part of z. Using (15), we obtain that

$$\left|e^{iuz}\right| \leq e^{|\operatorname{Im} z|} \tag{16}$$

for any $u \in [-1, 1]$, and any nonzero complex variable z.

Combining (5), (14), and (16) yields that

$$|J_{\nu+k}(z)| \leq \frac{2\, e^{|\operatorname{Im} z|}\, |z^\nu|\, |z|^k}{\sqrt{\pi}\; |2^\nu|\, 2^k\, |\Gamma(\nu+k+1/2)|} \tag{17}$$

for any nonzero complex variable z, and any integer k such that $k > |\nu| - 1$.

We now fix any integer n such that $n > |\nu| + \frac{1}{2}$. Combining (17) and (10) yields that

$$|J_{\nu+k}(z)| \leq B_{\nu,n}(z)\, \frac{|z|^{k-n}}{2^{k-n}\,(k-n)!} \tag{18}$$

for any nonzero complex variable z, and any integer k such that $k > n$, where

$$B_{\nu,n}(z) = \frac{2\, e^{|\operatorname{Im} z|}\, |z^\nu|\, |z|^n}{\sqrt{\pi}\; |2^\nu|\, 2^n\, |\Gamma(\nu+n+1/2)|} \tag{19}$$

for any nonzero complex variable z. Taking the limits of both sides of (18) as k tends toward ∞ yields the present theorem. □

4 Identities

This section provides the principal results of the present note.

The following theorem provides an analogue for Bessel functions of what is known as the Christoffel-Darboux identity for orthonormal polynomials.

Theorem 10 *Suppose that ν is a complex number.*

Then,

$$\sum_{k=1}^{\infty} 2\,(\nu+k)\; J_{\nu+k}(w)\; J_{\nu+k}(z) = \frac{wz}{w-z}\Big(J_{\nu+1}(w)\; J_\nu(z) - J_\nu(w)\; J_{\nu+1}(z)\Big) \tag{20}$$

for any distinct nonzero complex variables w and z.

Proof. Using (8), we obtain that

$$\sum_{k=1}^{n}\left(\frac{2\,(\nu+k)}{w}\; J_{\nu+k}(w)\right) J_{\nu+k}(z) - \sum_{k=1}^{n} J_{\nu+k}(w)\;\left(\frac{2\,(\nu+k)}{z}\; J_{\nu+k}(z)\right)$$
$$= J_\nu(w)\; J_{\nu+1}(z) - J_{\nu+1}(w)\; J_\nu(z) + J_{\nu+n+1}(w)\; J_{\nu+n}(z) - J_{\nu+n}(w)\; J_{\nu+n+1}(z) \tag{21}$$

for any distinct nonzero complex variables w and z, and any positive integer n; dividing both the left- and right-hand sides of (21) by $\frac{1}{w}-\frac{1}{z}$ and using Theorem 9 to take the limits as n tends toward ∞ yields (20). □

Remark 11 When ν is a nonnegative integer, (20), (4), and (3) provide a basis for the filtering and interpolation of linear combinations of Bessel functions on $(-\infty, \infty)$, as originated for orthonormal polynomials in [3] and [12], and subsequently optimized.

The following theorem states the limit of (20) as w tends to z.

Theorem 12 *Suppose that ν is a complex number.*

Then,

$$\sum_{k=1}^{\infty} 2\,(\nu+k)\;(J_{\nu+k}(z))^2 = z^2\left(J_\nu(z)\;\frac{d}{dz}J_{\nu+1}(z) - J_{\nu+1}(z)\;\frac{d}{dz}J_\nu(z)\right) \tag{22}$$

for any nonzero complex variable z.

Proof. Dividing both sides of (21) by $\frac{1}{w}-\frac{1}{z}$, we obtain that

$$\sum_{k=1}^{n} 2\,(\nu+k)\; J_{\nu+k}(w)\; J_{\nu+k}(z)$$
$$= \frac{wz}{w-z}\bigg(\Big(J_{\nu+1}(w) - J_{\nu+1}(z)\Big)\; J_\nu(z) - \Big(J_\nu(w) - J_\nu(z)\Big)\; J_{\nu+1}(z)\bigg)$$
$$+ \frac{wz}{w-z}\bigg(\Big(J_{\nu+n}(w) - J_{\nu+n}(z)\Big)\; J_{\nu+n+1}(z)$$
$$- \Big(J_{\nu+n+1}(w) - J_{\nu+n+1}(z)\Big)\; J_{\nu+n}(z)\bigg) \tag{23}$$

for any distinct nonzero complex variables w and z, and any positive integer n. Taking the limits of both sides of (23) as w tends to z, and then using (7), (8), and Theorem 9 to take the limits as n tends toward ∞, we obtain (22). □

The following theorem provides an alternative expression for the series in (22).

Theorem 13 *Suppose that ν is a complex number.*
Then,

$$\sum_{k=1}^{\infty} 2\,(\nu+k)\;(J_{\nu+k}(z))^2 = z^2\;(J_\nu(z))^2 + z^2\;(J_{\nu+1}(z))^2 - (2\nu+1)\,z\;J_\nu(z)\;J_{\nu+1}(z) \tag{24}$$

for any nonzero complex variable z.

Proof. Combining (22), (6), and (7) yields (24). □

The following theorem provides an expression for the sum of the terms with even indices in the series in (20).

Theorem 14 *Suppose that ν is a complex number.*
Then,

$$2\,(\nu+1)\sum_{k=1}^{\infty} 2\,(\nu+2k)\,J_{\nu+2k}(w)\;J_{\nu+2k}(z) = \frac{w^2\,z^2}{w^2-z^2}\Big(J_{\nu+2}(w)\;J_\nu(z) - J_\nu(w)\;J_{\nu+2}(z)\Big) \tag{25}$$

for any distinct nonzero complex variables w and z.

Proof. Using (9), we obtain that

$$\begin{aligned}
&2\,(\nu+1)\left(\sum_{k=1}^{n}\left(\frac{2\,(\nu+2k)}{w^2}\;J_{\nu+2k}(w)\right)\;J_{\nu+2k}(z) - \sum_{k=1}^{n} J_{\nu+2k}(w)\;\left(\frac{2\,(\nu+2k)}{z^2}\;J_{\nu+2k}(z)\right)\right)\\
&= J_\nu(w)\;J_{\nu+2}(z) - J_{\nu+2}(w)\;J_\nu(z) + \frac{\nu+1}{\nu+2n+1}\Big(J_{\nu+2n+2}(w)\;J_{\nu+2n}(z) - J_{\nu+2n}(w)\;J_{\nu+2n+2}(z)\Big)
\end{aligned} \tag{26}$$

for any distinct nonzero complex variables w and z, and any positive integer n, provided that ν is not an odd negative integer. When ν is an odd negative integer, (26) holds for any distinct nonzero complex variables w and z, and any sufficiently large integer n, by continuity from the cases when ν is not an odd negative integer. In all cases, dividing both sides of (26) by $\frac{1}{w^2} - \frac{1}{z^2}$ and using Theorem 9 to take the limits as n tends toward ∞ yields (25). □

Remark 15 Together, (25) and (4) provide a basis for the filtering and interpolation of linear combinations of Bessel functions on $(0,\infty)$, as originated for orthonormal polynomials in [3] and [12], and subsequently optimized.

5 Generalizations

This section derives an analogue, for any family of functions satisfying a symmetric "three-term" recurrence relation, of what is known as the Christoffel-Darboux identity for orthonormal polynomials.

Suppose that g and $\dots,\ f_{-2},\ f_{-1},\ f_0,\ f_1,\ f_2,\ \dots$ are complex-valued functions on a set S, and $\dots,\ c_{-2},\ c_{-1},\ c_0,\ c_1,\ c_2,\ \dots$, and $\dots,\ d_{-2},\ d_{-1},\ d_0,\ d_1,\ d_2,\ \dots$ are complex numbers, such that

$$g(x)\, f_k(x) = c_{k-1}\, f_{k-1}(x) + d_k\, f_k(x) + c_k\, f_{k+1}(x) \tag{27}$$

for any $x \in S$, and any integer k.

Using (27), we obtain that

$$\sum_{k=m+1}^{n} (g(x)\, f_k(x))\; f_k(y) - \sum_{k=m+1}^{n} f_k(x)\; (g(y)\, f_k(y)) = c_m \Big(f_m(x)\; f_{m+1}(y) - f_{m+1}(x)\; f_m(y) \Big) + c_n \Big(f_{n+1}(x)\; f_n(y) - f_n(x)\; f_{n+1}(y) \Big) \tag{28}$$

for any $x \in S$ and $y \in S$, and any integers m and n such that $m < n$.

Dividing both sides of (28) by $g(x) - g(y)$, we obtain that

$$\sum_{k=m+1}^{n} f_k(x)\; f_k(y) = \frac{c_m}{g(x) - g(y)} \Big(f_m(x)\; f_{m+1}(y) - f_{m+1}(x)\; f_m(y) \Big) + \frac{c_n}{g(x) - g(y)} \Big(f_{n+1}(x)\; f_n(y) - f_n(x)\; f_{n+1}(y) \Big) \tag{29}$$

for any $x \in S$ and $y \in S$ such that $g(x) \neq g(y)$, and any integers m and n such that $m < n$; (29) is analogous to the classical Christoffel-Darboux identity for orthonormal polynomials described, for example, in [5].

Section 4 implicitly applies (29) with $m = 0$ for the following choices of functions:

$$g(x) = \frac{1}{x} \tag{30}$$

and

$$f_k(x) = \sqrt{2\,(\nu + k)}\; J_{\nu+k}(x), \tag{31}$$

as well as

$$g(x) = \frac{1}{x^2} \tag{32}$$

and

$$f_k(x) = \sqrt{2\,(\nu + 2k)}\; J_{\nu+2k}(x). \tag{33}$$

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