Population Protocols

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June 10th, 2009

Acknowledgments

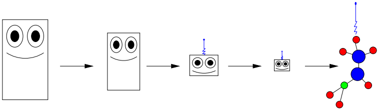
Joint work with:

- Dana Angluin (Yale)
- Melody Chan (Princeton)
- Zoë Diamadi (McKinsey & Company)
- David Eisenstat (Brown)
- Michael J. Fischer (Yale)
- Hong Jiang (Google)
- René Peralta (NIST)
- Eric Ruppert (York)

The past and future of computing

Economics of mass production push computer systems toward **large numbers** of **very limited** standardized components:

- Centralized systems
- Distributed systems
- Wireless distributed systems
- Sensor networks/RFID chips
- Smart molecules?

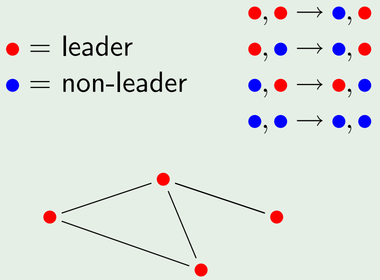


Our goal: take the limit of this process.

Population protocols

- A **population protocol** (Angluin, Aspnes, Diamadi, Fischer, and Peralta, PODC 2004) consists of a collection of **finite-state agents** organized in an **interaction graph**.
- An **interaction** between two neighbors updates the state of *both agents* according to a joint **transition function**.
- Interactions are *asymmetric*: one agent is the **initiator** and one the **responder**.

Leader Election



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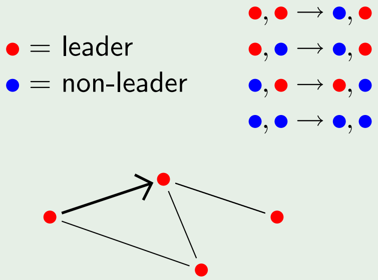

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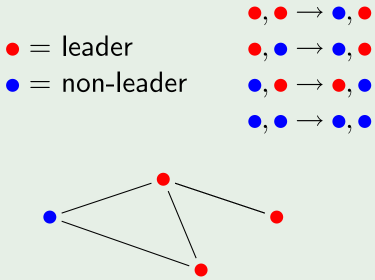

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● = non-leader

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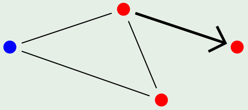

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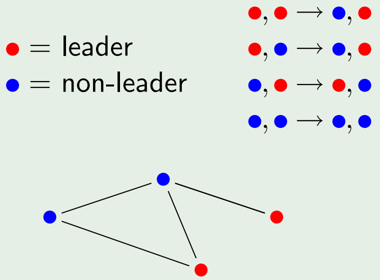

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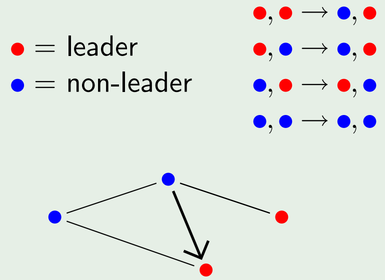

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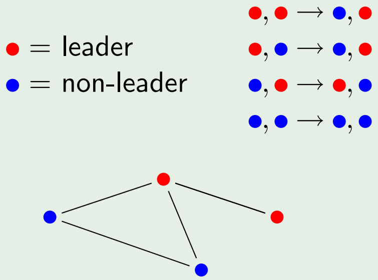

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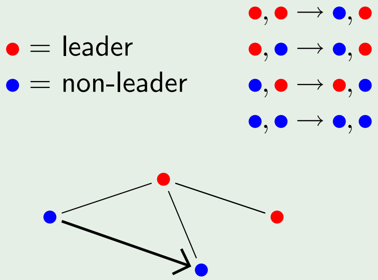

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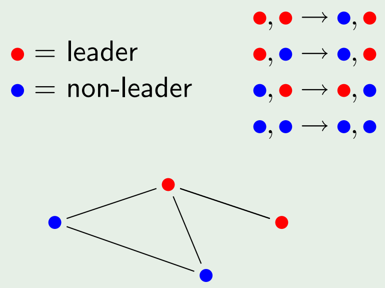

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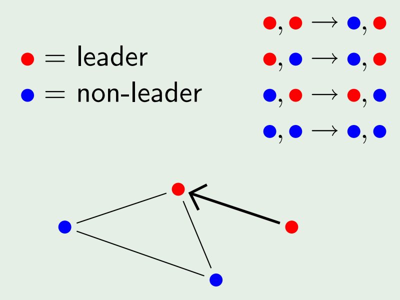

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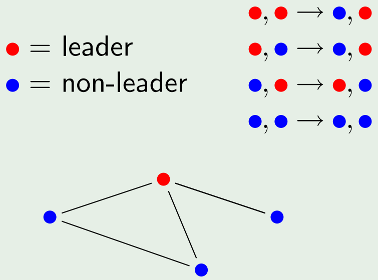

Stable computations

- **Input map** converts inputs (at each agent) to initial states.
- **Output map** extracts outputs from states.
- A **stable computation** converges to the same output at all agents.
- **Fairness condition** enforces that any reachable state is eventually reached.

Parity

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Out:

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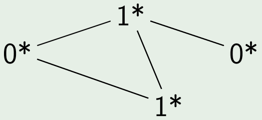

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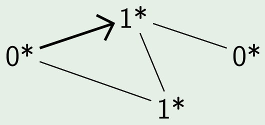

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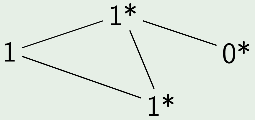

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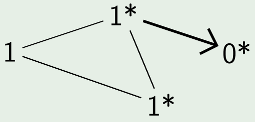

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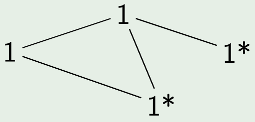

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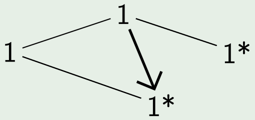

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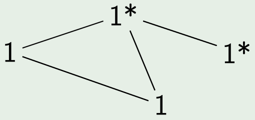

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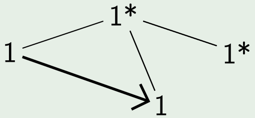

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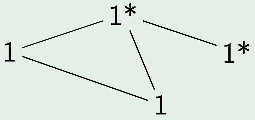

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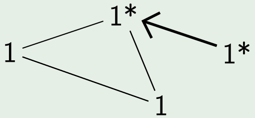

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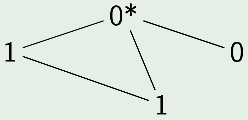

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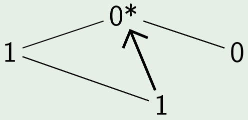

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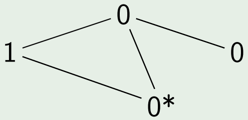

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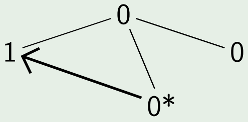

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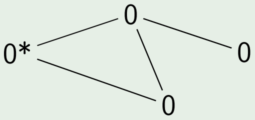

What we can compute

- Trick: represent numbers by tokens scattered across the population.
- Population protocols on connected graphs can **stably compute** all of **first-order Presburger arithmetic** on counts of input tokens, including
  - Addition.
  - Subtraction.
  - Multiplication by a constant k.
  - Remainder mod k.
  - $>$, $<$, and $=$.
  - $\wedge$, $\vee$, $\neg$, $\forall x$, and $\exists x$, applied to above.
- Example: "Are there at least twice as many 0 bits as 1 bits?"

Presburger predicates in disguise

Other ways to define a Presburger predicate:

- Take a regular language L and forget about the order of symbols in each word.
  - Resulting **Parikh map of a regular set** is Presburger-definable.
  - All Presburger-definable sets can be constructed this way.
  - Cute fact: going to context-free languages doesn't change anything.
- Take a finite union of **linear sets** of the form

$$\{\vec{b} + k_1\vec{x}_1 + k_2\vec{x}_2 + \cdots + k_m\vec{x}_m\}.$$

  - Resulting **semilinear set** is Presburger-definable.
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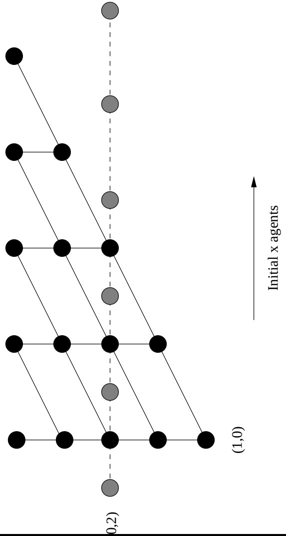
Example



Computability of Presburger predicates

- Computable for fixed inputs (Angluin *et al.*, PODC 2004)
- Computable if inputs converge after some finite time (Angluin, Aspnes, Chan, Fischer, Jiang, and Peralta, DCOSS 2005).
- Computable with one-way communication (Angluin, Aspnes, Eisenstat, Ruppert, OPODIS 2005).
- Computable if a small number of agents fail (Delporte-Gallet, Fauconnier, Guerraoui, Ruppert, DCOSS 2006).
- Nothing else is computable on a **complete interaction graph**, i.e. if any agent can interact with any other (Angluin, Aspnes, Eisenstat, PODC 2006).
    - Example: can't compute "Is the number of 0 bits the square of the number of 1 bits?"

Hooray! We're done!

- Question: If we have an exact characterization of what population protocols can do, aren't we done?
- Answer: No.

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- Question: If we have an exact characterization of what population protocols can do, aren't we done?
- Answer: No.
  - Bounded-degree interaction graph gives all of LINSPACE (Angluin *et al.*, DCOSS 2005).
  - Random scheduling in a complete graph gives all of LOGSPACE with exponential slowdown using simple techniques (Angluin et al., PODC 2004), or *polylogarithmic* slowdown using more sophisticated techniques (Angluin *et al.*, DISC 2006).
- Random scheduling + complete graph = test-tube full of molecules.

Randomized population protocols

- Assume next pair of agents to interact is chosen uniformly (i.e. with probability $\frac{1}{N(N-1)}$).
- This gives the **randomized population protocol** model from (Angluin *et al.*, PODC 2004).
- It also is equivalent to the uniform-rate case of the standard model for well-mixed chemical systems (e.g. (Gillespie, 1977)), **population processes** from the stochastic processes literature ((Kurtz, 1981)), and corresponds closely to the **stochastic chemical reaction networks** of (Soloveichik *et al.*, 2008).
- Expected **time** is obtained by dividing expected interactions by N—each agent interacts at a fixed rate regardless of size of the population.

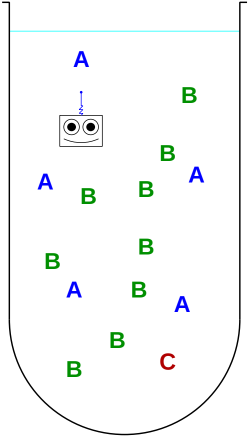
What does this have to do with DNA computing?

Agents	=	Molecules
Agent states	=	Species
Interactions	=	Reactions
Complete interaction graph	=	Well-mixed test tube
Uniform interaction rates	$\neq$	Varying reaction rates
Conservation of agents	$\neq$	Synthesis and decomposition

- Disclaimer: I just write transition tables, I don't know if they can be realized in a lab.
- For more chemically realistic models see (Soloveichik, Cook, Winfree, and Bruck, Computing with finite stochastic chemical reaction networks, Natural Computing 7(4):615–633, December 2008).

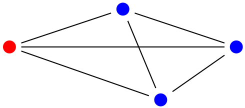
A test-tube computer

- **Register values** (up to $O(N)$) are stored as tokens distributed across the population.
- A unique **leader agent** acts as the (finite-state) CPU.
- We want to support the usual operations of addition, subtraction, comparison, multiplication, division, etc.
- We want to do them all in polylogarithmic time ($O(N \log^{O(1)} N)$ interactions).
- We'll accept a small ($O(N^{-\Theta(1)})$) probability of error.



Epidemics

- Key fact: An epidemic starting from one infected agent spreads to all agents in $\Theta(\log N)$ time with high probability.



- This gives us a broadcast primitive.

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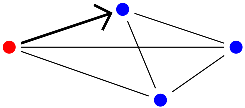

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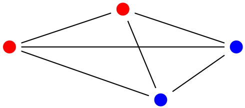

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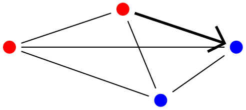

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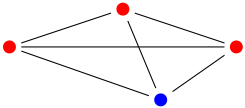

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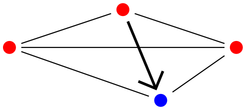

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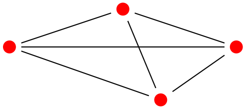

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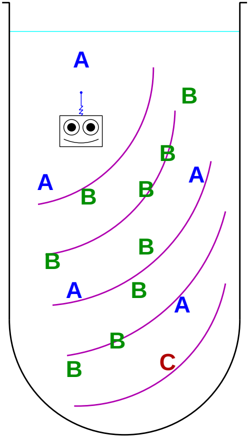
Instruction cycle

- Leader propagates a new opcode via epidemic.
- Followers carry out chosen operation:
  - $A \leftarrow 0$: Erase your A token upon receipt of opcode.
  - $A \leftarrow A + B$: Make a new A token for each B token.
  - $A \stackrel{?}{=} 0$: Start a counter-epidemic if you have an A.
  - $A > B$, $A \leftarrow A - B$, etc.: more complicated.
- Leader collects response (if any) from counter-epidemic, updates its state, and starts a new cycle.



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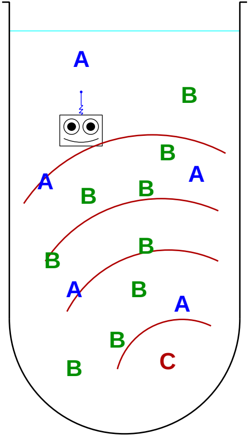
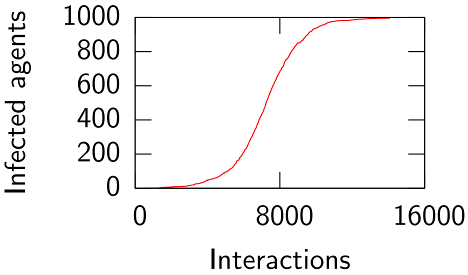


What's missing?

Problem: How does the leader know when to start the next instruction cycle?

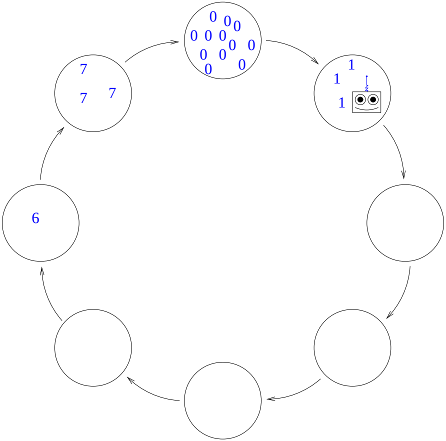
Bounding the time for epidemics



- Average interactions to infect next victim is $\frac{N(N-1)}{i(N-i)}$.
- For $i > N/2$, this is $\Theta(N/i)$, the waiting time for coupon collector.
- $\Rightarrow$ Known coupon collector concentration results (Kamath *et al.*, 1995) bound $i > N/2$ case: $\Theta(N \log N)$ w.h.p.
- Symmetry bounds $i > N/2$ case.

Phase clock

- Each agent is in a **phase** in the range 0 to $m - 1$.
- An initiator in a later phase $\bmod m$ recruits agents in earlier phases.
- The leader advances if it sees an initiator in its own phase.
- Result: Leader goes all the way around every $\Theta(\log N)$ time units.



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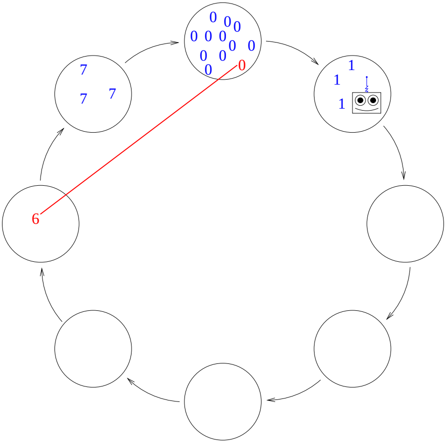

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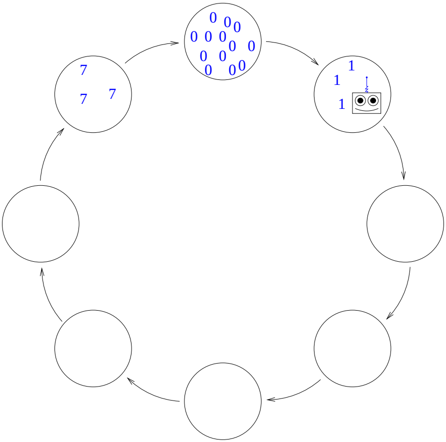

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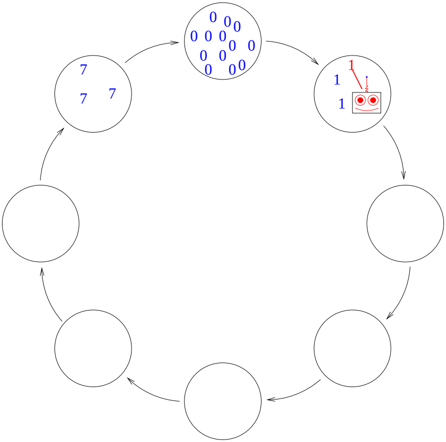

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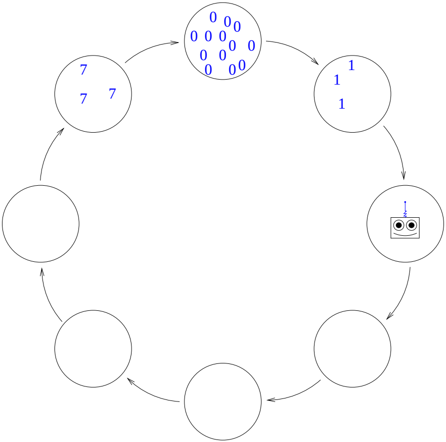
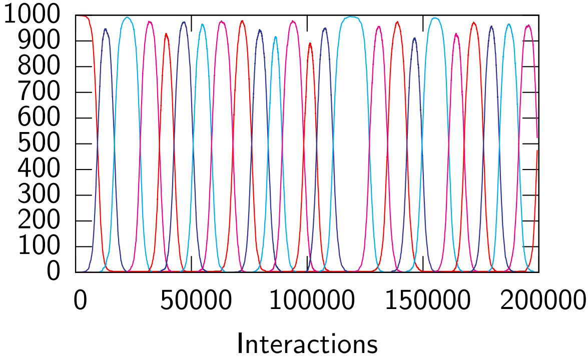
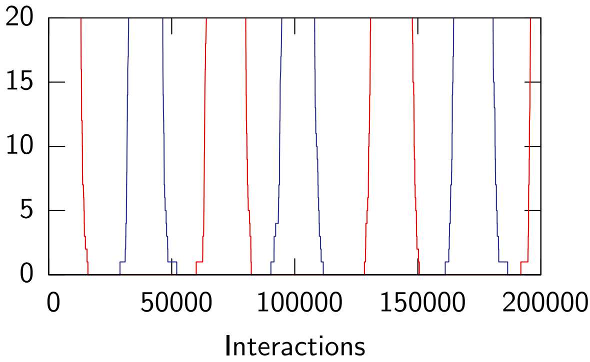


Phase clock: simulation results



Phase clock with $N = 1000$ and $m = 8$.

Phase clock: simulation results



Zoomed view of phase 0 and phase 4.

Why it works

- Phases i and higher act as an epidemic wiping out phases $i-1$ and lower.
- This epidemic finishes in $a \log N$ time (with high probability).
- When the leader advances, it takes at least $b \log N$ time (w.h.p.) to generate at least N^{ϵ} agents in the same phase $\Rightarrow$ leader advances before $b \log N$ time (a **short phase**) with probability $N^{O(\epsilon)-1}$.
- For a sufficiently large number of phases m, the chance of too many short phases in a row is $O(N^{-c})$.
- Amazing fact: m depends on c but not N.

Other operations

- Operations like assignment and addition that don't require tokens to interact can be done in one instruction cycle ($O(\log N)$ time).
- Operations that do require interaction may take longer.
  - Naive $A \overset{?}{>} B$ algorithm: Have A and B tokens cancel until only one kind is left.
  - This takes $\Omega(N^2)$ interactions if there are few A's and B's.
- How can we do cancellation faster?

Cancellation by amplification

- Cancellation is fast if there are many tokens to cancel.
- Solution: Alternate between canceling and doubling.
- Invariant $A_k - B_k = 2^k(A_0 - B_0)$ after k rounds.
- If no winner in $2 \log N$ rounds, $A_0 = B_0$.
- This gives $A \stackrel{?}{<} B$ in $O(\log^2 N)$ time.

Subtraction and division by binary search

- To compute $C \leftarrow A - B$, do binary search for C such that $A = B + C$.
- This takes $O(\log N)$ rounds of binary search at $O(\log^2 N)$ time each $\Rightarrow O(\log^3 N)$ time.
- Similar approach for division gives $O(\log^5 N)$ time. (This is our most expensive operation.)

Results

For a randomized population protocol with a unique initial leader, we have:

- Register machine simulation:
  - $\Theta(\log N)$-bit registers.
  - $O(\log^5 N)$ expected time per operation. ($O(\log N)$ in later work.)
  - $O(N^{-c})$ probability of failure.
- Presburger predicate computation:
  - $O(\log^5 N)$ expected time. (Cf. $O(N)$ for previous protocols.)
  - Zero probability of failure.
  - Trick: Combine fast fallible protocol with slow robust one.

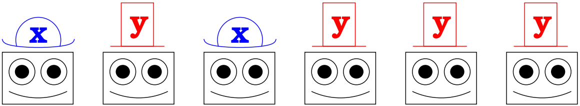
Why $O(\log^5 N)$?

- Main problem: Comparisons take too long.
- Solution: See next slide.

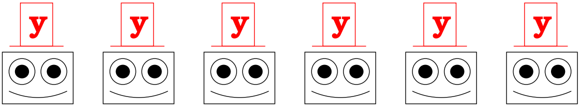
Fast robust approximate majority

(Angluin, Aspnes, and Eisenstat, DISC 2007).

1. Start with mixed population of x and y agents:

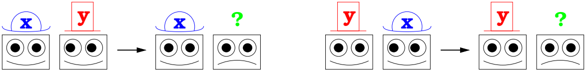


2. Run for $O(\log N)$ time.
3. Obtain (with high probability) majority value everywhere:



Confusion creates blank agents

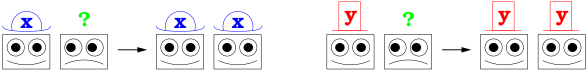
- Three states: x, y, and b (blank).
- If I see disagreement, I go blank:



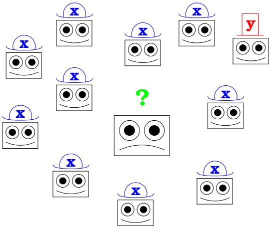
- Equally likely to remove an x or a y.
- Never removes last non-blank token.

Fashion favors the majority

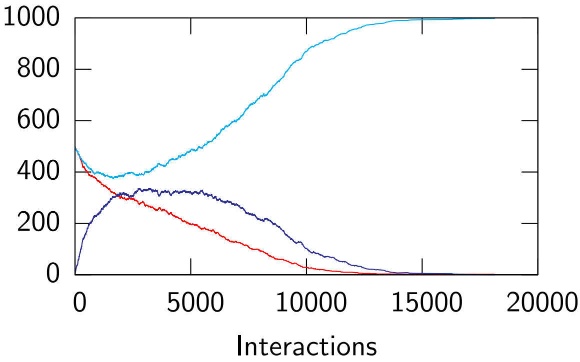
- Blank agents adopt whatever value they see:



- Favors more common value ⇒ pushes towards unanimity.

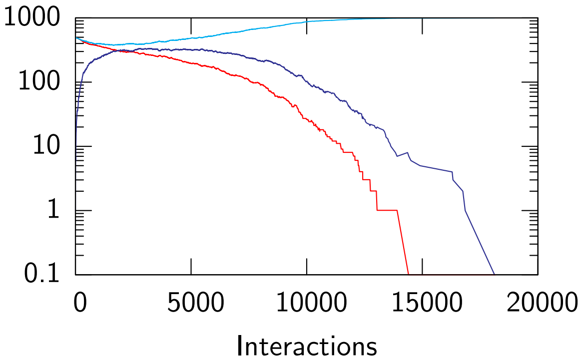


Simulation results



Approximate majority starting with $x = y = 500$.

Simulation results (log scale)



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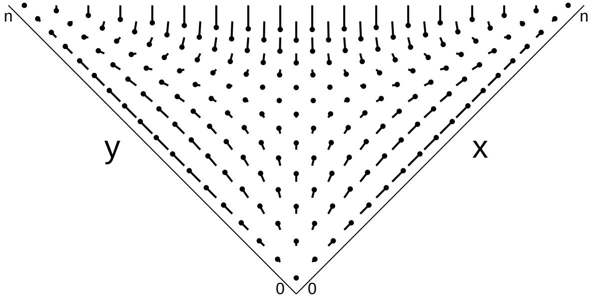
Configuration space



- Transitions push to $y = N$ and $x = N$ corners.
- Unstable equilibrium at $y = x = b$.
- Want to show $O(N \log N)$ bound on steps to convergence.

The proof I wish we had

One thing to try:

- Use standard results on limits of population processes (Kurtz, Wormald) to get system of differential equations.
- Solve them to find convergence bounds in the limit as $N \to \infty$.

But it doesn't work:

- Known limit results only work up to $O(N)$ interactions.
- Small ($o(1)$) concentrations of agents go to 0 in the limit.
- Resulting differential equations don't have nice solutions anyway.

What we did instead

- Use a potential function to bound number of xb and yb interactions.
- Basic idea: track $u = x - y$.
  - When $|u|$ is small, acts like random walk: u^2 rises on average.
  - When $|u|$ is big, acts like exponential growth: $\log |u|$ rises on average.
  - Compromise: $f = \log\left(\frac{3}{2}N + u^2\right)$ acts like u^2 for small $|u|$ and $\log |u|$ for large $|u|$.
- Bound xy and yx interactions by conservation of agents.
- This leaves xx, yy, and bb interactions, but these are rare except in the corners.
- Separate potential functions cover corner cases.
- Final result: $O(N \log N)$ steps with high probability.

Correctness

Majority value is correct if the initial margin is $\omega(\sqrt{N \log N})$

- Couple (u_i) with an unbiased random walk (t_i) so that $|t_i| \leq |u_i|$
  - $\Pr[u \text{ increases}] \geq 1/2$ for $u \geq 0$
  - $\Pr[u \text{ decreases}] \geq 1/2$ for $u \leq 0$
- Suppose $t_0 = u_0 = x_0 - y_0 = \omega(\sqrt{N \log N})$
- With high probability, random walk t_i is positive for $\Theta(N \log N)$ steps $\Rightarrow x$ wins.
- Argue symmetrically for y.

This even works if $o(\sqrt{N})$ agents are **Byzantine**, meaning they can pretend to have any value in any interaction.

Application

- Previous register machine simulation
- $+$ fast comparison operation
- $+$ some other tricks
- $=$ $O(\log N)$-time register machine operations.
- This is optimal.

Can we build it?

- I can't, but maybe somebody here can.
- Fast robust approximate majority is both simple and fault-resistant.
- Other protocols are more elaborate and more brittle.

Can we analyze it?

- Brute force analysis works, but isn't pretty.
- Can't even analyze majority with 3 non-blank token types.
- Better tools are needed.
- But ability to do computation limits what we can do.

More information:

http://www.cs.yale.edu/homes/aspnes/introduction-to-population-protocols-abstract.html.