Elements of auction theory

This material is not part of the course, but is included here for those who are interested

Overview

- Some connections among auctions
- Efficiency and revenue maximization
- Incentive compatibility (or truth revelation)
  - 2nd price auction
  - The generalized Vickrey auction (GVA)
- The revelation principle
- Independent private value versus common value
- Some results on revenue maximization
- Deriving the Nash eq. in 1st-price auction

Some connections among auctions

- Dutch = 1st price sealed bid
- English ~ Japanese
- English ~ 2nd price sealed bid (both price setting and expected revenue under IPV model; see below)

Two yardsticks for good auctions

• Revenue: The seller should extract the highest possible price

• Efficiency: The buyer with the highest valuation should get the good

The two are usually aligned, but can sometimes fail (e.g, 1st-price auction when buyers have different risk attitudes).

Direct mechanisms and incentive compatibility

- In a direct mechanism you simply announce your valuation
- The auction is incentive compatible if it's in your best interest not to lie about your true valuation
- Example: 2nd price ("Vickrey") auction
- Another example: the generalized Vickrey auction (GVA); see discussion of combinatorial auctions below

The revelation principle

- You can transform any auction into an “equivalent” one which is direct and incentive compatible
- “Rather than lie, the mechanism will lie for you”
- Example: Assume two bidders, with valuations drawn uniformly from a fixed interval (plus other assumptions). The optimal strategy is to bid 1/2 your true value. But if the rule is changed so that the winner only pays half his bid, it is optimal to bid your true value.

Various auction settings

- Independent Private Value (IPV)
- Common Value (CV)
- Affiliated Value

These different settings define different Bayesian games; the relationships between the private signals determine the auction settings

Some results on revenue maximization

- IPV model
  - Risk-neutral buyers: all k-price auctions, as well as English (Japanese) and Dutch have identical revenues.
  - Risk-Averse buyers: 1st + Dutch are preferable to 2nd and English; more generally, k-price is superior to k+1 price.
  - Risk-seeking buyers: k+1 price is preferable to k price (e.g., English is preferable to Dutch)
- CV model
  - English and second-price are no longer revenue equivalent, although Dutch and 1st still are
  - English is preferable to 2nd, which is preferable to 1st/Dutch, if the agents are risk-neutral.

Example of equilibirum analysis: 1st-price auction

- Setting:
  - One good, two agents
  - The agents' valuations are independently drawn from the uniform distribution on [0,1]
  - $u(y)=y$ is the utility functions of both agents
  - A first-price (FP) auction

Easy part: proving a particular equilibrium

- Assume player 1 plays z, and player 2's strategy is b(y)=y/2
- If player 1's valuation is x his expected payoff is given by

$$\int_{v=0}^{2z} (x-z)dv = (x-z)2z = 2xz - 2z^2$$

(note: given the y/2 strategy, 1 only wins when 2's valuation is <2z)

- This is a quadratic equation who derivative $2x - 4z$ is equal to 0 at $z = x/2$
- The same analysis is true of player 2
- Therefore b(x)=x/2 is the "best response" to the same strategy by the other player, and therefore the two players adopting this strategy forms an equilibrium

Harder part: proving this equilibrium unique, under certain assumptions (sketch)*

- We'll be looking for a continuous symmetric increasing equilibrium.
- 1's expected payoff is

$$\int_{v=0}^{b^{-1}(z)} (x-z)\,dv = b^{-1}(z)(x-z)$$

- Since this is a concave fn (requires proof), max is achieved when derivative is zero:

* This is not part of the class material, and is included for completeness

Second approach (cont.)

$$b^{-1}(z)(x-z)'*b^{-1}(z)'(x-z)=$$

$$b^{-1}(z)+\frac{(x-(b(x)))}{b'(x)}=\frac{-b^{-1}(z)b'(x)+x-b(x)}{b'(x)}$$

(take this ↗ part on faith, even if you didn't see this in your intro calculus course)

- We now look for a value for z which zeros the derivative, under the constraint that z=b(x):

$$-b^{-1}(b(x))b'(x)+x-b(x)=xb'(x)+x-b(x)=0$$

- Now note that b(x)=x/2 is a solution; and using functional analysis, which we won't enter, you can prove it is the unique solution

Multi-player 1st-price auctions

- More generally, with n bidders and similar conditions, the symmetric equilibrium is given by:

$$b(x) = \frac{n-1}{n} x$$

Some of the many topics in auction theory not covered here

- "signals" and valuations: the formal model of auctions as Bayesian games
- Precise definition of the "affiliated values" model
- Monopoly, marginal revenue, marginal cost
- Risk-averse auctioneer (auctioneer as trader): if bidders are risk neutral auctioneer prefers $1^{st} > 2^{nd} >$ English(Japanese).
- Constant (absolute) risk aversion
- Computational complexity of combinatorial auctions
- Collusion in auctions

Further reading on auction theory

- *Handbook of Auction Theory (P. Klemperer, Ed.)*
- *Klemperer's introduction to the Handbook*
- *Articles references in the Handbook*
- *Many other, more recent articles*