2. Overview and Background

This is a course for computer system designers and builders, and for people who want to really understand how systems work, especially concurrent, distributed, and fault-tolerant systems.

The course teaches you
 how to write precise specifications for any kind of computer system,
 what it means for code to satisfy a specification, and
 how to prove that it does.
It also shows you how to use the same methods less formally, and gives you some suggestions for deciding how much formality is appropriate (less formality means less work, and often a more understandable spec, but also more chance to overlook an important detail).

The course also teaches you a lot about the topics in computer systems that we think are the most important: persistent storage, concurrency, naming, networks, distributed systems, transactions, fault tolerance, and caching. The emphasis is on
 careful specifications of subtle and sometimes complicated things,
 the important ideas behind good code, and
 how to understand what makes them actually work.
We spend most of our time on specific topics, but we use the general techniques throughout. We emphasize the ideas that different kinds of computer system have in common, even when they have different names.

The course uses a formal language called Spec for writing specs and code; you can think of it as a very high level programming language. There is a good deal of written introductory material on Spec (explanations and finger exercises) as well as a reference manual and a formal semantics. We introduce Spec ideas in class as we use them, but we do not devote class time to teaching Spec per se; we expect you to learn it on your own from the handouts. The one to concentrate on is handout 3, which has an informal introduction to the main features and lots of examples. Section 9 of handout 4, the reference manual, should also be useful. The rest of the reference manual is for reference, not for learning. Don't overlook the one page summary at the end of handout 3.

Because we write specs and do proofs, you need to know something about logic. Since many people don't, there is a concise treatment of the logic you will need at the end of this handout.

This is not a course in computer architecture, networks, operating systems, or databases. We will not talk in detail about how to code pipelines, memory interconnects, multiprocessors, routers, data link protocols, network management, virtual memory, scheduling, resource allocation, SQL, relational integrity, or TP monitors, although we will deal with many of the ideas that underlie these mechanisms.

Topics

General

Specifications as state machines.
The Spec language for describing state machines (writing specs and code).
What it means to implement a spec.
Using abstraction functions and invariants to prove that a program implements a spec.

What it means to have a crash.
What every system builder needs to know about performance.

Specific

Disks and file systems.
Practical concurrency using mutexes (locks) and condition variables; deadlock.
Hard concurrency (without locking): models, specs, proofs, and examples.
Transactions: simple, cached, concurrent, distributed.
Naming: principles, specs, and examples.
Distributed systems: communication, fault-tolerance, and autonomy.
Networking: links, switches, reliable messages and connections.
Remote procedure call and network objects.
Fault-tolerance, availability, consensus and replication.
Caching and distributed shared memory.

Previous editions of the course have also covered security (authentication, authorization, encryption, trust) and system management, but this year we are omitting these topics in order to spend more time on concurrency and semantics and to leave room for project presentations.

Prerequisites

There are no formal prerequisites for the course. However, we assume some knowledge both of computer systems and of mathematics. If you have taken 6.033 and 6.042, you should be in good shape. If you are missing some of this knowledge you can pick it up as we go, but if you are missing a lot of it you can expect to have serious trouble. It's also important to have a certain amount of maturity: enough experience with systems and mathematics to feel comfortable with the basic notions and to have some reliable intuition.

If you know the meaning of the following words, you have the necessary background. If a lot of them are unfamiliar, this course is probably not for you.

Systems

Cache, virtual memory, page table, pipeline
Process, scheduler, address space, priority
Thread, mutual exclusion (locking), semaphore, producer-consumer, deadlock
Transaction, commit, availability, relational data base, query, join
File system, directory, path name, striping, RAID
LAN, switch, routing, connection, flow control, congestion
Capability, access control list, principal (subject)

If you have not already studied Lampson's paper on hints for system design, you should do so as background for this course. It is Butler Lampson, Hints for computer system design, *Proceedings of the Ninth ACM Symposium on Operating Systems Principles*, October 1983, pp 33-48. There is a pointer to it on the course Web page.

Programming

Invariant, precondition, weakest precondition, fixed point
Procedure, recursion, stack
Data type, sub-type, type-checking, abstraction, representation

Object, method, inheritance
Data structures: list, hash table, binary search, B-tree, graph

Mathematics

Function, relation, set, transitive closure
Logic: proof, induction, de Morgan's laws, implication, predicate, quantifier
Probability: independent events, sampling, Poisson distribution
State machine, context-free grammar
Computational complexity, unsolvable problem

If you haven't been exposed to formal logic, you should study the summary at the end of this handout.

References

These are places to look when you want more information about some topic covered or alluded to in the course, or when you want to follow current research. You might also wish to consult Prof. Saltzer's bibliography for 6.033, which you can find on the course web page.

Books

Some of these are fat books better suited for reference than for reading cover to cover, especially Cormen, Leiserson, and Rivest, Jain, Mullender, Hennessy and Patterson, and Gray and Reuter. But the last two are pretty easy to read in spite of their encyclopedic character.

Specification: Leslie Lamport, *Specifying Systems: The TLA+ Language and Tools for Hardware and Software Engineers,* Addison-Wesley, 2002. TLA+ is superficially quite different from Spec, but the same underneath. Lamport's approach is somewhat more mathematical than ours, but in the same spirit. You can find this book at http://research.microsoft.com/users/lamport/tla/book.html.

Systems programming: Greg Nelson, ed., *Systems Programming with Modula-3*, Prentice-Hall, 1991. Describes the language, which has all the useful features of C++ but is much simpler and less error-prone, and also shows how to use it for concurrency (a version of chapter 4 is a handout in this course), an efficiently customizable I/O streams package, and a window system.

Performance: Jon Bentley, *Writing Efficient Programs*, Prentice-Hall, 1982. Short, concrete, and practical. Raj Jain, *The Art of Computer Systems Performance Analysis*, Wiley, 1991. Tells you much more than you need to know about this subject, but does have a lot of realistic examples.

Algorithms and data structures: Robert Sedgwick, *Algorithms*, Addison-Wesley, 1983. Short, and usually tells you what you need to know. Tom Cormen, Charles Leiserson, and Ron Rivest, *Introduction to Algorithms*, McGraw-Hill, 1989. Comprehensive, and sometimes valuable for that reason, but usually tells you a lot more than you need to know.

Distributed algorithms: Nancy Lynch, *Distributed Algorithms*, Morgan Kaufmann, 1996. The bible for distributed algorithms. Comprehensive, but a much more formal treatment than in this course. The topic is algorithms, not systems.

Computer architecture: John Hennessy and David Patterson, *Computer Architecture: A Quantitative Approach*, 2nd edition, Morgan Kaufmann, 1995. The bible for computer architecture.

The second edition has lots of interesting new material, especially on multiprocessor memory systems and interconnection networks. There's also a good appendix on computer arithmetic; it's useful to know where to find this information, though it has nothing to do with this course.

Transactions, data bases, and fault-tolerance: Jim Gray and Andreas Reuter, *Transaction Processing: Concepts and Techniques*, Morgan Kaufmann, 1993. The bible for transaction processing, with much good material on data bases as well; it includes a lot of practical information that doesn't appear elsewhere in the literature.

Networks: Radia Perlman, *Interconnections: Bridges and Routers*, Addison-Wesley, 1992. Not exactly the bible for networking, but tells you nearly everything you might want to know about how packets are actually switched in computer networks.

Distributed systems: Sape Mullender, ed., *Distributed Systems*, 2nd ed., Addison-Wesley, 1993. A compendium by many authors that covers the field fairly well. Some chapters are much more theoretical than this course. Chapters 10 and 11 are handouts in this course. Chapters 1, 2, 8, and 12 are also recommended. Chapters 16 and 17 are the best you can do to learn about real-time computing; unfortunately, that is not saying much.

User interfaces: Alan Cooper, *About Face*, IDG Books, 1995. Principles, lots of examples, and opinionated advice, much of it good, from the original designer of Visual Basic.

Journals

You can find all of these in the CSAIL reading room in 32-G882. The cryptic strings in brackets are call numbers there. You can also find the ACM publications in the ACM digital library at www.acm.org.

For the current literature, the best sources are the proceedings of the following conferences. 'Sig' is short for "Special Interest Group", a subdivision of the ACM that deals with one field of computing. The relevant ones for systems are SigArch for computer architecture, SigPlan for programming languages, SigOps for operating systems, SigComm for communications, SigMod for data bases, and SigMetrics for performance measurement and analysis.

Symposium on Operating Systems Principles (SOSP; published as special issues of ACM SigOps *Operating Systems Review*; fall of odd-numbered years) [P4.35.06]

Operating Systems Design and Implementation (OSDI; Usenix Association, now published as special issues of ACM *SigOps Review*; fall of even-numbered years, except spring 1999 instead of fall 1998) [P4.35.U71]

Architectural Support for Programming Languages and Operating Systems (ASPLOS; published as special issues of ACM SigOps *Operating Systems Review*, SigArch *Computer Architecture News*, or *SigPlan Notices*; fall of even-numbered years) [P6.29.A7]

Applications, Technologies, Architecture, and Protocols for Computer Communication, (SigComm conference; published as special issues of ACM SigComm *Computer Communication Review*; annual) [P6.24.D31]

Principles of Distributed Computing (PODC; ACM; annual) [P4.32.D57]

Very Large Data Bases (VLDB; Morgan Kaufmann; annual) [P4.33.V4]

International Symposium on Computer Architecture (ISCA; published as special issues of ACM SigArch *Computer Architecture News*; annual) [P6.20.C6]

Less up to date, but more selective, are the journals. Often papers in these journals are revised versions of papers from the conferences listed above.

ACM Transactions on Computer Systems

ACM Transactions on Database Systems

ACM Transactions on Programming Languages and Systems

There are often good survey articles in the less technical IEEE journals:

IEEE *Computer*, *Networks*, *Communication*, *Software*

The Internet Requests for Comments (RFC's) can be reached from

`http://www.cis.ohio-state.edu/hypertext/information/rfc.html`

Rudiments of logic

Propositional logic

The basic type is `Bool`, which contains two elements `true` and `false`. Expressions in these operators (and the other ones introduced later) are called 'propositions'.

Basic operators. These are ∧ (and), ∨ (or), and ~ (not).[1] The meaning of these operators can be conveniently given by a 'truth table' which lists the value of `a op b` for each possible combination of values of `a` and `b` (the operators on the right are discussed later) along with some popular names for certain expressions and their operands.

		negation	conjunction	disjunction	equality		implication
		not	and	or			implies
a	b	~a	a ∧ b	a ∨ b	a = b	a ≠ b	a ⇒ b
T	T	F	T	T	T	F	T
T	F		F	T	F	T	F
F	T	T	F	T	F	T	T
F	F		F	F	T	F	T
name of a			conjunct	disjunct			antecedent
name of b			conjunct	disjunct			consequent

Note: In Spec we write ==> instead of the ⇒ that mathematicians use for implication. Logicians write ⊃ for implication, which looks different but is shaped like the > part of ⇒.

Since the table has only four rows, there are only 16 Boolean operators, one for each possible arrangement of `T` and `F` in a column. Most of the ones not listed don't have common names, though 'not and' is called 'nand' and 'not or' is called 'nor' by logic designers.

The ∧ and ∨ operators are
commutative and
associative and
distribute over each other.
That is, they are just like * (times) and + (plus) on integers, except that + doesn't distribute over *:

`a + (b * c) ≠ (a + b) * (a + c)`

but ∨ does distribute over ∧:

`a ∨ (b ∧ c) = (a ∨ b) ∧ (a ∨ c)`

An operator that distributes over ∧ is called 'conjunctive'; one that distributes over ∨ is called 'disjunctive'. Both ∧ and ∨ are both conjunctive and disjunctive. This takes some getting used to.

The relation between these operators and ~ is given by DeMorgan's laws (sometimes called the "bubble rule" by logic designers), which say that you can push ~ inside ∧ or ∨ (or pull it out) by flipping from one to the other:

```
~ (a ∧ b)  =  ~a ∨ ~b
~ (a ∨ b)  =  ~a ∧ ~b
```

[1] It's possible to write all three in terms of the single operator 'nor' or 'nand', but our goal is clarity, not minimality.

To put a complex expression into " disjunctive normal form" replace terms in = and ⇒ with their equivalents in ∧, ∨, and ~ (given below), use DeMorgan's laws to push all the ~'s in past ∧ and ∨ so that they apply to variables, and then distribute ∧ over ∨ so that the result looks like

$$(a_1 \wedge \sim a_2 \wedge \ldots) \vee (\sim b_1 \wedge b_2 \wedge \ldots) \vee \ldots$$

The disjunctive normal form is unique (up to ordering, since ∧ and ∨ are commutative). Of course, you can also distribute ∨ over ∧ to get a unique "conjunctive normal form".

If you want to find out whether two expressions are equal, one way is to put them both into disjunctive (or conjunctive) normal form, sort the terms, and see whether they are identical. Another way is to list all the possible values of the variables (if there are *n* variables, there are 2^n of them) and tabulate the values of the expressions for each of them; we saw this 'truth table' for some two-variable expressions above.

Because `Bool` is the result type of relations like `=`, you can write expressions that mix up relations with other operators in ways that are impossible for any other type. Notably

$$\boxed{(a = b) \;=\; ((a \wedge b) \vee (\sim a \wedge \sim b))}$$

Some people feel that the outer = in this expression is somehow different from the inner one, and write it ≡. Experience suggests, however, that this is often a harmful distinction to make.

Implication. We can define an ordering on `Bool` with `false > true`, that is, `false` is greater than `true`. The non-strict version of this ordering is called 'implication' and written ⇒ (rather than ≥ or `>=` as we do with other types; logicians write it ⊃, which also looks like an ordering symbol). So $(\mathtt{true} \Rightarrow \mathtt{false}) = \mathtt{false}$ (read this as: "`true` is greater than or equal to `false`" is false) but all other combinations are `true`. The expression $a \Rightarrow b$ is pronounced "`a` implies `b`", or "if `a` then `b`".[2]

There are lots of rules for manipulating expressions containing ⇒; the most useful ones are given below. If you remember that ⇒ is an ordering you'll find it easy to remember most of the rules, but if you forget the rules or get confused, you can turn the ⇒ into ∨ by the rule

$$\boxed{(a \Rightarrow b) \;=\; \sim a \vee b}$$

and then just use the simpler rules for ∧, ∨, and ~. So remember this even if you forget everything else.

The point of implication is that it tells you when one proposition is stronger than another, in the sense that if the first one is true, the second is also true (because if both `a` and $a \Rightarrow b$ are `true`, then `b` must be `true` since it can't be `false`).[3] So we use implication all the time when reasoning from premises to conclusions. Two more ways to pronounce $a \Rightarrow b$ are "`a` is stronger than `b`" and "`b` follows from `a`". The second pronunciation suggests that it's sometimes useful to write the operands in the other order, as $b \Leftarrow a$, which can also be pronounced "`b` is weaker than `a`" or "`b` only if `a`"; this should be no surprise, since we do it with other orderings.

[2] It sometimes seems odd that `false` implies `b` regardless of what `b` is, but the "if ... then" form makes it clearer what is going on: if `false` is `true` you can conclude anything, but of course it isn't. A proposition that implies `false` is called 'inconsistent' because it implies anything. Obviously it's bad to think that an inconsistent proposition is true. The most likely way to get into this hole is to think that each of a collection of innocent looking propositions is true when their conjunction turns out to be inconsistent.

[3] It may also seem odd that `false > true` rather than the other way around, since `true` seems better and so should be bigger. But in fact if we want to conclude lots of things, being close to `false` is better because if `false` is true we can conclude anything, but knowing that `true` is true doesn't help at all. Strong propositions are as close to `false` as possible; this is logical brinkmanship. For example, $a \wedge b$ is closer to `false` than `a` (there are more values of the variables `a` and `b` that make it `false`), and clearly we can conclude more things from it than from `a` alone.

Of course, implication has the properties we expect of an ordering:

Transitive: If $a \Rightarrow b$ and $b \Rightarrow c$ then $a \Rightarrow c$.[4]

Reflexive: $a \Rightarrow a$.

Anti-symmetric: If $a \Rightarrow b$ and $b \Rightarrow a$ then $a = b$.[5]

Furthermore, ~ reverses the sense of implication (this is called the 'contrapositive'):

$$(a \Rightarrow b) \;=\; (\sim b \Rightarrow \sim a)$$

More generally, you can move a disjunct on the right to a conjunct on the left by negating it, or vice versa. Thus

$$(a \Rightarrow b \vee c) \;=\; (a \wedge \sim b \Rightarrow c)$$

As special cases in addition to the contrapositive we have

$$(a \Rightarrow b) \;=\; (a \wedge \sim b \Rightarrow \mathtt{false}) \;=\; \sim(a \wedge \sim b) \vee \mathtt{false} = \sim a \vee b$$

$$(a \Rightarrow b) \;=\; (\mathtt{true} \Rightarrow \sim a \vee b) \;=\; \mathtt{false} \vee \sim a \vee b = \sim a \vee b$$

since `false` and `true` are the identities for ∨ and ∧.

We say that an operator `op` is 'monotonic' in an operand if replacing that operand with a stronger (or weaker) one makes the result stronger (or weaker). Precisely, "`op` is monotonic in its first operand" means that if $a \Rightarrow b$ then $(a \ \mathtt{op}\ c) \Rightarrow (b \ \mathtt{op}\ c)$. Both ∧ and ∨ are monotonic; in fact, any operator that is conjunctive (distributes over ∧) is monotonic, because if $a \Rightarrow b$ then $a = (a \wedge b)$, so

$$a \ \mathtt{op}\ c = (a \wedge b) \ \mathtt{op}\ c = a \ \mathtt{op}\ c \wedge b \ \mathtt{op}\ c \Rightarrow b \ \mathtt{op}\ c$$

If you know what a lattice is, you will find it useful to know that the set of propositions forms a lattice with ⇒ as its ordering and (remember, think of ⇒ as "greater than or equal"):

top	= `false`		
bottom	= `true`		
meet	= ∧	least upper bound,	so $(a \wedge b) \Rightarrow a$ and $(a \wedge b) \Rightarrow b$
join	= ∨	greatest lower bound,	so $a \Rightarrow (a \vee b)$ and $b \Rightarrow (a \vee b)$

This suggests two more expressions that are equivalent to $a \Rightarrow b$:

$(a \Rightarrow b) = (a = (a \wedge b))$ — 'and'ing a weaker term makes no difference, because $a \Rightarrow b$ iff $a =$ least upper bound(`a`, `b`).

$(a \Rightarrow b) = (b = (a \vee b))$ — 'or'ing a stronger term makes no difference, because $a \Rightarrow b$ iff $b =$ greatest lower bound(`a`, `b`).

Predicate logic

Propositions that have free variables, like $x < 3$ or $x < 3 \Rightarrow x < 5$, demand a little more machinery. You can turn such a proposition into one without a free variable by substituting some value for the variable. Thus if $P(x)$ is $x < 3$ then $P(5)$ is $5 < 3 = \mathtt{false}$. To get rid of the free variable without substituting a value for it, you can take the 'and' or 'or' of the proposition for all the possible values of the free variable. These have special names and notation[6]:

$\forall\, x \mid P(x) \;=\; P(x_1) \wedge P(x_2) \wedge \ldots$ — for all `x`, `P(x)`. In Spec, `(ALL x | P(x))` or $\wedge : \{x \mid P(x)\}$

[4] We can also write this $((a \Rightarrow b) \wedge (b \Rightarrow c)) \Rightarrow (a \Rightarrow c)$.

[5] Thus $(a = b) = (a \Rightarrow b \wedge b \Rightarrow a)$, which is why $a = b$ is sometimes pronounced "`a` if and only if `b`" and written "`a` iff `b`".

[6] There is no agreement on what symbol should separate the $\forall\, x$ or $\exists\, x$ from the $P(x)$. We use '|' here as Spec does, but other people use '.' or ':' or just a space, or write $(\forall\, x)$ and $(\exists\, x)$. Logicians traditionally write (x) and $(\exists\, x)$.

$\exists$ x | P(x) = P(x_1) $\vee$ P(x_2) $\vee$ ... there exists an x such that P(x). In Spec, (EXISTS x | P(x)) or $\vee$: {x | P(x)}

Here the x_i range over all the possible values of the free variables.[7] The first is called 'universal quantification'; as you can see, it corresponds to conjunction. The second is called 'existential quantification' and corresponds to disjunction. If you remember this you can easily figure out what the quantifiers do with respect to the other operators.

In particular, DeMorgan's laws generalize to quantifiers:

```
~ (∀ x | P(x))  =  (∃ x | ~P(x))
~ (∃ x | P(x))  =  (∀ x | ~P(x))
```

Also, because $\wedge$ and $\vee$ are conjunctive and therefore monotonic, $\forall$ and $\exists$ are conjunctive and therefore monotonic.

It is not true that you can reverse the order of $\forall$ and $\exists$, but it's sometimes useful to know that having $\exists$ first is stronger:

```
∃ y | ∀ x | P(x, y)  ⇒  ∀ x | ∃ y | P(x, y)
```

Intuitively this is clear: a y that works for every x can surely do the job for each particular x.

If we think of P as a relation, the consequent in this formula says that P is total (relates every x to some y). It doesn't tell us anything about how to find a y that is related to x. As computer scientists, we like to be able to compute things, so we prefer to have a function that computes y, or the set of y's, from x. This is called a 'Skolem function'; in Spec you write P.func (or P.setF for the set). P.func is total if P is total. Or, to turn this around, if we have a total function f such that $\forall$ x | P(x, f(x)), then certainly $\forall$ x | $\exists$ y | P(x, y); in fact, y = f(x) will do. Amazing.

Summary of logic

The $\wedge$ and $\vee$ operators are commutative and associative and distribute over each other.

DeMorgan's laws: ~ (a $\wedge$ b) = ~a $\vee$ ~b
~ (a $\vee$ b) = ~a $\wedge$ ~b

Any expression has a unique (up to ordering) disjunctive normal form in which $\vee$ combines terms in which $\vee$ combines (possibly negated) variables: (a_1 $\wedge$ ~a_2 $\wedge$ ...) $\vee$ (~b_1 $\wedge$ b_2 $\wedge$ ...) $\vee$ ...

Implication: (a $\Rightarrow$ b) = ~a $\vee$ b

Implication is the ordering in a lattice (a partially ordered set in which every subset has a least upper and a greatest lower bound) with

top	= false	so false $\Rightarrow$ true
bottom	= true	
meet	= $\wedge$	least upper bound, so (a $\wedge$ b) $\Rightarrow$ a
join	= $\vee$	greatest lower bound, so a $\Rightarrow$ (a $\vee$ b)

For all x, P(x):

$\forall$ x | P(x) = P(x_1) $\wedge$ P(x_2) $\wedge$ ...

There exists an x such that P(x):

$\exists$ x | P(x) = P(x_1) $\vee$ P(x_2) $\vee$ ...

[7] In general this might not be a countable set, so the conjunction and disjunction are written in a somewhat misleading way, but this complication won't make any difference to us.

Index for logic

3. Introduction to Spec

This handout explains what the Spec language is for, how to use it effectively, and how it differs from a programming language like C, Pascal, Clu, Java, or Scheme. Spec is very different from these languages, but it is also much simpler. Its meaning is clearer and Spec programs are more succinct and less burdened with trivial details. The handout also introduces the main constructs that are likely to be unfamiliar to a programmer. You will probably find it worthwhile to read it over more than once, until those constructs are familiar. Don't miss the one-page summary of spec at the end. The handout also has an index.

Spec is a language for writing precise descriptions of digital systems, both sequential and concurrent. In Spec you can write something that differs from practical code (for instance, code written in C) only in minor details of syntax. This sort of thing is usually called a program. Or you can write a very high level description of the behavior of a system, usually called a specification. A good specification is almost always quite different from a good program. You can use Spec to write either one, but not the same *style* of Spec. The flexibility of the language means that you need to know the purpose of your Spec in order to write it well.

Most people know a lot more about writing programs than about writing specs, so this introduction emphasizes how Spec differs from a programming language and how to use it to write good specs. It does not attempt to be either complete or precise, but other handouts fill these needs. The *Spec Reference Manual* (handout 4) describes the language completely; it gives the syntax of Spec precisely and the semantics informally. *Atomic Semantics of Spec* (handout 9) describes precisely the meaning of an atomic command; here 'precisely' means that you should be able to get an unambiguous answer to any question. The section "Non-Atomic Semantics of Spec" in handout 17 on formal concurrency describes the meaning of a non-atomic command.

Spec's notation for commands, that is, for changing the state, is derived from Edsger Dijkstra's guarded commands (E. Dijkstra, *A Discipline of Programming*, Prentice-Hall, 1976) as extended by Greg Nelson (G. Nelson, A generalization of Dijkstra's calculus, *ACM TOPLAS* **11**, 4, Oct. 1989, pp 517-561). The notation for expressions is derived from mathematics.

This handout starts with a discussion of specifications and how to write them, with many small examples of Spec. Then there is an outline of the Spec language, followed by three extended examples of specs and code. At the end are two handy tear-out one-page summaries, one of the language and one of the official POCS strategy for writing specs and code.

In the language outline, the parts in small type describe less important features, and you can skip them on first reading.

What is a specification for?

The purpose of a specification is to communicate precisely all the essential facts about the behavior of a system. The important words in this sentence are:

communicate	The spec should tell both the client and the implementer what each needs to know.
precisely	We should be able to prove theorems or compile machine instructions based on the spec.
essential	Unnecessary requirements in the spec may confuse the client or make it more expensive to implement the system.
behavior	We need to know exactly what we mean by the behavior of the system.

Communication

Spec mediates communication between the client of the system and its implementer. One way to view the spec is as a contract between these parties:

> The client agrees to depend only on the system behavior expressed in the spec; in return it only has to read the spec, and it can count on the implementer to provide a system that actually does behave as the spec says it should.
>
> The implementer agrees to provide a system that behaves according to the spec; in return it is free to arrange the internals of the system however it likes, and it does not have to deliver anything not laid down in the spec.

Usually the implementer of a spec is a programmer, and the client is another programmer. Usually the implementer of a program is a compiler or a computer, and the client is a programmer.

Usually the system that the implementer provides is called an implementation, but in this course we will call it *code* for short. It doesn't have to be C or Java code; we will give lots of examples of code in Spec which would still require a lot of work on the details of data structures, memory allocation, etc. to turn it into an executable program. You might wonder what good this kind of high-level code is. It expresses the difficult parts of the design clearly, without the straightforward details needed to actually make it run.

Behavior

What do we mean by behavior? In real life a spec defines not only the functional behavior of the system, but also its performance, cost, reliability, availability, size, weight, etc. In this course we will deal with these matters informally if at all. The Spec language doesn't help much with them.

Spec is concerned only with the possible state transitions of the system, on the theory that the possible state transitions tell the complete story of the functional behavior of a digital system. So we make the following definitions:

> A *state* is the values of a set of names (for instance, `x=3, color=red`).

A *history* is a sequence of states such that each pair of adjacent states is a transition of the system (for instance, `x=1; x=2; x=5` is the history if the initial state is `x=1` and the transitions are "if `x = 1` then `x := x + 1`" and "if `x = 2` then `x := 2 * x + 1`").

A *behavior* is a set of histories (a non-deterministic system can have more than one history, usually at least one for every possible input).

How can we specify a behavior?

One way to do this is to just write down all the histories in the behavior. For example, if the state just consists of a single integer, we might write

```
 1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1
 1  2  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1
...
 1  2  1  2  1  2  1  2  1  2  1  2  1  2  1  2  1  2
....
 1  2  3  4  5  1  2  3  1  2  3  4  5  6  7  8  9 10
```

The example reveals two problems with this approach:

The sequences are long, and there are a lot of them, so it takes a lot of space to write them down. In fact, in most cases of interest the sequences are infinite, so we can't actually write them down.

It isn't too clear from looking at such a set of sequences what is really going on.

Another description of this set of sequences from which these examples are drawn is "18 integers, each one either `1` or one more than the preceding one." This is concise and understandable, but it is not formal enough either for mathematical reasoning or for directions to a computer.

Precise

In Spec the set of sequences can be described in many ways, for example, by the expression

```
{q: SEQ Int |    q.size = 18
             /\ (ALL i: Int | 0 <= i /\ i < q.size ==>
                                q(i) = 1 \/ (i > 0 /\ q(i) = q(i-1) + 1)) }
```

Here the expression in `{...}` is very close to the usual mathematical notation for defining a set. Read it as "The set of all `q` which are sequences of integers such that `q.size = 18` and ...". Spec sequences are indexed from `0`. The `(ALL ...)` is a universally quantified predicate, and `==>` stands for implication, since Spec uses the more familiar `=>` for 'then' in a guarded command. Throughout Spec the '`|`' symbol separates a declaration of some new names and their types from the scope in which they are meaningful.

Alternatively, here is a state machine that generates the sequences we want. We specify the transitions of the machine by starting with primitive *assignment commands* and putting them together with a few kinds of compound commands. Each command specifies a set of possible transitions.

```
VAR i, j |
     << i := 1; j := 1 >> ;
     DO << j < 18 => BEGIN i := 1 [] i := i+1 END; Output(i); j := j+1 >> OD
```

Here there is a good deal of new notation, in addition to the familiar semicolons, assignments, and plus signs.

`VAR i, j |` introduces the local variables `i` and `j` with arbitrary values. Because `;` binds more tightly than `|`, the scope of the variables is the rest of the example.

The `<< ... >>` brackets delimit the atomic actions or transitions of the state machine. All the changes inside these brackets happen as one transition of the state machine.

`j < 18 => ...` is a transition that can only happen when `j < 18`. Read it as "if `j < 18` then `...`". The `j < 18` is called a *guard*. If the guard is false, we say that the entire command *fails*.

`i := 1 [] i := i + 1` is a *non-deterministic* transition which can either set `i` to `1` or increment it. Read `[]` as 'or'.

The `BEGIN ... END` brackets are just brackets for commands, like { ... } in C. They are there because `=>` binds more tightly than the `[]` operator inside the brackets; without them the meaning would be "either set `i` to `1` if `j < 18` or increment `i` and `j` unconditionally".

Finally, the `DO ... OD` brackets mean: repeat the `...` transition as long as possible. Eventually `j` becomes `18` and the guard becomes false, so the command inside the `DO ... OD` fails and can no longer happen.

The expression approach is better when it works naturally, as this example suggests, so Spec has lots of facilities for describing values: sequences, sets, and functions as well as integers and booleans. Usually, however, the sequences we want are too complicated to be conveniently described by an expression; a state machine can describe them much more easily.

State machines can be written in many different ways. When each transition involves only simple expressions and changes only a single integer or boolean state variable, we think of the state machine as a program, since we can easily make a computer exhibit this behavior. When there are transitions that change many variables, non-deterministic transitions, big values like sequences or functions, or expressions with quantifiers, we think of the state machine as a spec, since it may be much easier to understand and reason about it, but difficult to make a computer exhibit this behavior. In other words, large atomic actions, non-determinism, and expressions that compute sequences or functions are hard to code. It may take a good deal of ingenuity to find code that has the same behavior but uses only the small, deterministic atomic actions and simple expressions that are easy for the computer.

Essential

The hardest thing for most people to learn about writing specs is that *a spec is not a program*. A spec defines the behavior of a system, but unlike a program it need not, and usually should not, give any practical method for producing this behavior. Furthermore, it should pin down the behavior of the system only enough to meet the client's needs. Details in the spec that the client doesn't need can only make trouble for the implementer.

The example we just saw is too artificial to illustrate this point. To learn more about the difference between a spec and code consider the following:

```
CONST eps := 10**-8

APROC SquareRoot0(x: Real) -> Real =
    << VAR y : Real | Abs(x - y*y) < eps => RET y >>
```

(Spec as described in the reference manual doesn't have a `Real` data type, but we'll add it for the purpose of this example.)

The combination of `VAR` and `=>` is a very common Spec idiom; read it as "choose a `y` such that `Abs(x - y*y) < eps` and do `RET y`". Why is this the meaning? The `VAR` makes a choice of any `Real` as the value of `y`, but the entire transition on the second line cannot occur unless the guard `Abs(x - y*y) < eps` is true. Hence the `VAR` must choose a value that satisfies the guard.

What can we learn from this example? First, the result of `SquareRoot0(x)` is not completely determined by the value of `x`; any result whose square is within `eps` of `x` is possible. This is why `SquareRoot0` is written as a procedure rather than a function; the result of a function has to be determined by the arguments and the current state, so that the value of an expression like `f(x) = f(x)` will be `true`. In other words, `SquareRoot0` is *non-deterministic*.

Why did we write it that way? First of all, there might not be any `Real` (that is, any floating-point number of the kind used to represent `Real`) whose square exactly equals `x`. We could accommodate this fact of life by specifying the closest floating-point number.[1] Second, however, we may not want to pay for code that gives the closest possible answer. Instead, we may settle for a less accurate answer in the hope of getting the answer faster.

You have to make sure you know what you are doing, though. This spec allows a negative result, which is perhaps not what we really wanted. We could have written (highlighting changes with boxes):

```
APROC SquareRoot1(x: Real) -> Real =
    << VAR y : Real | y >= 0 /\ Abs(x - y*y) < eps => RET y >>
```

to rule that out. Also, the spec produces no result if `x < 0`, which means that `SquareRoot1(-1)` will fail (see the section on commands for a discussion of failure). We might prefer a total function that raises an exception:

```
APROC SquareRoot2(x: Real) -> Real RAISES {undefined} =
    <<    x >= 0 => VAR y : Real | y >= 0 /\ Abs(x - y*y) < eps => RET y
      [*]  RAISE undefined >>
```

The `[*]` is 'else'; it does its second operand iff the first one fails. Exceptions in Spec are much like exceptions in CLU. An exception is contagious: once started by a `RAISE` it causes any containing expression or command to yield the same exception, until it runs into an exception handler (not shown here). The `RAISES` clause of a routine declaration must list all the exceptions that the procedure body can generate, either by `RAISES` or by invoking another routine.

Code for this spec would look quite different from the spec itself. Instead of the existential quantifier implied by the `VAR y`, it would have an algorithm for finding `y`, for instance, Newton's method. In the algorithm you would only see operations that have obvious codes in terms of the load, store, arithmetic, and test instructions of a computer. Probably the code would be deterministic.

Another way to write these specs is as functions that return the set of possible answers. Thus

```
FUNC SquareRoots1(x: Real) -> SET Real =
    RET {y : Real | y >= 0 /\ Abs(x - y*y) < eps}
```

[1] This would still be non-deterministic in the case that two such numbers are equally close, so if we wanted a deterministic spec we would have to give a rule for choosing one of them, for instance, the smaller.

Note that the form inside the `{...}` set constructor is the same as the guard on the `RET`. To get a single result you can use the set's `choose` method: `SquareRoots1(2).choose`.[2]

In the next section we give an outline of the Spec language. Following that are three extended examples of specs and code for fairly realistic systems. At the end is a one-page summary of the language.

An outline of the Spec language

The Spec language has two main parts:

- An *expression* describes how to compute a result (a value or an exception) as a function of other values: either literal constants or the current values of state variables.
- A *command* describes possible transitions of the state variables. Another way of saying this is that a command is a relation on states: it allows a transition from `s1` to `s2` iff it relates `s1` to `s2`.

Both are based on the *state*, which in Spec is a mapping from names to values. The names are called state variables or simply variables: in the sequence example above they are `i` and `j`. Actually a command relates states to *outcomes*; an outcome is either a state (a normal outcome) or a state together with an exception (an exceptional outcome).

There are two kinds of commands:

- An *atomic* command describes a set of possible transitions, or equivalently, a set of pairs of states, or a relation between states. For instance, the command `<< i := i + 1 >>` describes the transitions `i=1`→`i=2`, `i=2`→`i=3`, etc. (Actually, many transitions are summarized by `i=1`→`i=2`, for instance, `(i=1, j=1)`→`(i=2, j=1)` and `(i=1, j=15)`→`(i=2, j=15)`). If a command allows more than one transition from a given state we say it is non-deterministic. For instance, on page 3 the command `BEGIN i := 1 [] i := i + 1 END` allows the transitions `i=2`→`i=1` and `i=2`→`i=3`, with the rest of the state unchanged.
- A *non-atomic* command describes a set of *sequences* of states (by contrast with the set of pairs for an atomic command). More on this below.

A sequential program, in which we are only interested in the initial and final states, can be described by an atomic command.

The meaning of an expression, which is a function from states to values (or exceptions), is much simpler than the meaning of an atomic command, which is a relation between states, for two reasons:

- The expression yields a single value rather than an entire state.
- The expression yields at most one value, whereas a non-deterministic command can yield many final states.

[2] `r := SquareRoots1(x).choose` (using the function) is almost the same as `r := SquareRoot1(x)` (using the procedure). The difference is that because `choose` is a function it always returns the same element (even though we don't know in advance which one) when given the same set, and hence when `SquareRoots1` is given the same argument. The procedure, on the other hand, is non-deterministic and can return different values on successive calls, so that spec is weaker. Which one is more appropriate?

An atomic command is still simple, because its meaning is just a relation between states. The relation may be partial: in some states there may be no way to execute the command. When this happens we say that the command is not *enabled* in those states. As we saw, the relation is not necessarily a function, since the command may be non-deterministic.

A non-atomic command is much more complicated than an atomic command, because:

- Taken in isolation, the meaning of a non-atomic command is a relation between an initial state and a history. A history is a whole sequence of states, much more complicated than a single final state. Again, many histories can stem from a single initial state.
- The meaning of the (parallel) composition of two non-atomic commands is not any simple combination of their relations, such as the union, because the commands can interact if they share any variables that change.

These considerations lead us to describe the meaning of a non-atomic command by breaking it down into its atomic subcommands and connecting these up with a new state variable called a program counter. The details are somewhat complicated; they are sketched in the discussion of atomicity below, and described in handout 17 on formal concurrency.

The moral of all this is that you should use the simpler parts of the language as much as possible: expressions rather than atomic commands, and atomic commands rather than non-atomic ones. To encourage this style, Spec has a lot of syntax and built-in types and functions that make it easy to write expressions clearly and concisely. You can write many things in a single Spec expression that would require a number of C statements, or even a loop. Of course, code with a lot of concurrency will necessarily have more non-atomic commands, but this complication should be put off as long as possible.

Organizing the program

In addition to the expressions and commands that are the core of the language, Spec has four other mechanisms that are useful for organizing your program and making it easier to understand.

- A *routine* is a named computation with parameters, in other words, an abstraction of the computation. Parameters are passed by value. There are four kinds of routine:

  A *function* (defined with `FUNC`) is an abstraction of an expression.

  An *atomic procedure* (defined with `APROC`) is an abstraction of an atomic command.

  A general procedure (defined with `PROC`) is an abstraction of a non-atomic command.

  A *thread* (defined with `THREAD`) is the way to introduce concurrency.

- A *type* is a highly stylized assertion about the set of values that a name or expression can assume. A type is also a convenient way to group and name a collection of routines, called its *methods*, that operate on values in that set.
- An *exception* is a way to report an unusual outcome.
- A *module* is a way to structure the name space into a two-level hierarchy. An identifier `i` declared in a module `m` has the name `m.i` throughout the program. A *class* is a module that can be instantiated many times to create many objects, much like a Java class.

A Spec program is some global declarations of variables, routines, types, and exceptions, plus a set of modules each of which declares some variables, routines, types, and exceptions.

The next two sections describe things about Spec's expressions and commands that may be new to you. They should be enough for the Spec you will read and write in this course, but they don't answer every question about Spec; for those answers, read the reference manual and the handouts on Spec semantics.

Paragraphs in small print contain material that you might want to skip on first reading.

There is a one-page summary of the Spec language at the end of this handout.

Expressions, types, and relations

Expressions are for computing functions of the state.

A Spec expression is	*and its value is*
a constant	the constant
a variable	the current value of the variable
an invocation of a function on an argument that is some sub-expression	the value of the function at the value of the argument

There are no side-effects; those are the province of commands. There is quite a bit of syntactic sugar for function invocations. An expression may be undefined in a state; if a simple command evaluates an undefined expression, the command fails (see below).

Types

A Spec type defines two things:

A set of values; we say that a value *has* the type if it's in the set. The sets are not disjoint. If `T` is a type, `T.all` is its set of values.

A set of functions called the *methods* of the type. There is convenient syntax `v.m` for invoking method `m` on a value `v` of the type. A method `m` of type `T` is lifted to a method `m` of a set of `T`'s, a function `U->T`, or a relation from `U` to `T` in the obvious way, by applying it to the set elements or the result of the function or relation, unless overridden by a different `m` in the definition of the higher type. Thus if `int` has a `square` method, `{2, 3, 4}.square = {4, 9, 16}`. We'll see that this is a form of function composition.

Spec is strongly typed. This means that you are supposed to declare the types of your variables, just as you do in Java. In return the language defines a type for every expression[3] and ensures that the value of the expression always has that type. In particular, the value of a variable always has the declared type. You should think of a type declaration as a stylized comment that has a precise meaning and can be checked mechanically.

If `Foo` is a type, you can omit it in a declaration of the identifiers `foo, foo1, foo'` etc. Thus

```
VAR int1, bool2, char' | ...
```

[3] Note that a value may have many types, but a variable or an expression has exactly one type: for a variable, it's the declared type, and for a complex expression it's the result type of the top-level function in the expression.

is short for

```
VAR int1: Int, bool2: Bool, char': Char | ...
```

Note that this can be confusing in a declaration like `t, u: Int`, where `u` has type `U`, not type `Int`.

If `e IN T.all` then `e AS T` is an expression with the same value and type `T`; otherwise it's undefined. You can write `e IS T` for `e IN T.all`.

Spec has the usual types:

- `Int`, `Nat` (non-negative `Int`), `Bool`
- sets `SET T`
- functions `T->U`
- relations `T->>U`
- records or structs `[f1: T1, f2: T2, ...]`
- tuples `(T1, T2, ...)`
- variable-length arrays called sequences, `SEQ T`

A sequence is actually a function whose domain is `{0, 1, ..., n-1}` for some `n`. A record is actually a function whose domain is the field names, as strings. In addition to the usual functions like `"+"` and `"\/"`, Spec also has some less usual operations on these types, which are valuable when you want to suppress code detail; they are called constructors and combinations and are described below.

You can make a type with fewer values using `SUCHTHAT`. For example,

```
TYPE T = Int SUCHTHAT 0 <= t /\ t <= 4
```

has the value set `{0, 1, 2, 3, 4}`. Here the expression following `SUCHTHAT` is short for `(\ t: Int | 0 <= t /\ t <= 4)`, a lambda expression (with `\` for λ) that defines a function from `Int` to `Bool`, and a value has type `T` if it's an `Int` and the function maps it to `true`. You can write `this` for the argument of `SUCHTHAT` if the type doesn't have a name. The type `IN s`, where `s` has type `SET T`, is short for `SET T SUCHTHAT this IN s`.

Methods

Methods are a convenient way of packaging up some functions with a type so that the functions can be applied to values of that type concisely and without mentioning the type itself. For example, if `s` is a `SEQ T`, `s.head` is `(Sequence[T].Head)(s)`, which is just `s(0)` (which is undefined if `s` is empty). You can see that it's shorter to write `s.head`.[4] Note that when you write `e.m`, the method `m` is determined by the static type of `e`, and *not* by the value as in most object-oriented languages.

You can define your own methods by using `WITH`. For instance, consider

```
TYPE Complex = [re: Real, im: Real] WITH {"+":=Add, mag:=Mag}
```

The `[re: Real, im: Real]` is a record type (a `struct` in C) with fields `re` and `im`. `Add` and `Mag` are ordinary Spec functions that you must define, but you can now invoke them on a `c` which is `Complex` by writing `c + c'` and `c.mag`, which mean `Add(c, c')` and `Mag(c)`. You can use existing operator symbols or make up your own; see section 3 of the reference manual for lexical rules. You can also write `Complex."+"` and `Complex.mag` to denote the functions `Add` and `Mag`; this may be convenient if `Complex` was declared in a different module. Using `Add` as a method does not make it private, hidden, static, local, or anything funny like that.

[4] Of course, `s(0)` is shorter still, but that's an accident; there is no similar alternative for `s.tail`.

When you nest `WITH` the methods pile up in the obvious way. Thus

```
TYPE MoreComplex = Complex WITH {"-":=Sub, mag:=Mag2}
```

has an additional method `"-"`, the same `"+"` as `Complex`, and a different `mag`. Many people call this 'inheritance' and 'overriding'.

A method `m` of type `T` is *lifted* automatically to a method of types `V->T`, `V->>T`, and `SET T` by composing it with the value of the higher-order type. This is explained in detail in the discussion of functions below.

Expressions

The syntax for expressions gives various ways of writing function invocations in addition to the familiar `f(x)`. You can use unary and binary operators, and you can invoke a method with `e1.m(e2)` for `T.m(e1,e2)`, of just `e.m` if there are no other arguments. You can also write a lambda expression `(\ t: T | e)` or a conditional expression `(predicate => e1 [*] e2)`, which yields `e1` if `predicate` is true and `e2` otherwise. If you omit `[*] e2`, the result is undefined if `predicate` is false. Because `=>` denotes if ... then, implication is written `==>`.

Here is a list of all the built-in operators, which also gives their precedence, and a list of the built-in methods. You should read these over so that you know the vocabulary. The rest of this section explains many of these and gives examples of their use.

Note that any lattice (any partially ordered set with least upper bound or max, and greatest lower bound or min, defined on any pair of elements) has operators `/\` (max) and `\/` (min). Booleans, sets, and relations are examples of lattices. Any totally ordered set such as `Int` is a lattice.

Binary operators

Op	*Prec.*	*Argument/result types*	*Operation*
`**`	8	`(Int, Int)->Int`	exponentiate
`*`	7	`(Int, Int)->Int`	multiply
		`(T->U, U->V)->(T->V)`	function or relation composition: (\t | e_2(e_1(t)))
`/`	7	`(Int, Int)->Int`	divide
`//`	7	`(Int, Int)->Int`	remainder
`+`	6	`(Int, Int)->Int`	add
		`(SEQ T, SEQ T)->SEQ T`	concatenation
		`(T->U, T->U)->(T->U)`	function overlay: (\t | (e_2!t => e_2(t) [*] e_1(t))
`-`	6	`(Int, Int)->Int`	subtract
		`(SET T, SET T)->SET T`	set difference
		`(SEQ T, SEQ T)->SEQ T`	multiset difference
`!`	6	`(T->U, T)->Bool`	function is defined at arg
`!!`	6	`(T->U, T)->Bool`	function defined, no exception at arg
`..`	5	`(Int, Int)->SEQ Int`	subrange: {e_1, e_1+1, ..., e_2}
`||`	5	`(SEQ T, SEQ U)->SEQ(T,U)`	zip: pair of sequences to sequence of pairs
`<=`	4	`(Int, Int)->Bool`	less than or equal
		`(SET T, SET T)->Bool`	subset
		`(SEQ T, SEQ T)->Bool`	prefix: e_2.restrict(e_1.dom) = e_1
`<`	4	`(T, T)->Bool, T with <=`	less than
`>`	4	`(T, T)->Bool, T with <=`	greater than
`>=`	4	`(T, T)->Bool, T with <=`	greater or equal
`=`	4	`(Any, Any)->Bool`	can't override by `WITH`
`#`	4	`(Any, Any)->Bool`	not equal; can't override by `WITH`
`<<=`	4	`(SEQ T, SEQ T)->Bool`	non-contiguous sub-seq: (∃s | s<=e_2.dom ∧ s.sort*e_2=e_1)
`IN`	4	`(T, SET T)->Bool`	membership
`/\`	2	`(Bool, Bool)->Bool`	conditional and*

		`(T, T)->T`	max, for any lattice; example: set/relation intersection
`\/`	1	`(Bool, Bool)->Bool`	conditional or*
		`(T, T)->T`	min, for any lattice; example: set/relation union
`==>`	0	`(Bool, Bool)->Bool`	conditional implies*
`op`	5	`(T, U)->V`	`op` none of the above: `T."op"(e1, e2)`

The "*" on the conditional Boolean operators means that, unlike all other operators, they don't evaluate their second argument if the first one determines the result. Thus `f(x) /\ g(x)` is `false` if `f(x)` is false, even if `g(x)` is undefined.

Unary operators

Op	*Prec.*	*Argument/result types*	*Operation*
`-`	6	`Int->Int`	negation
`~`	3	`Bool->Bool`	complement
		`SET T->SET T`	set complement
		`(T->>U)->(T->>U)`	relation complement
`op`	5	`T->U`	`op` none of the above: `T."op"(e1)`

Relations

A relation `r` is a generalization of a function: an arbitrary set of ordered pairs, defined by a predicate, a total function from pairs to `Bool`. Thus `r` can relate an element of its domain to any number of elements of its range (including none). Like a function, `r` has `dom`, `rng`, and `inv` methods (the inverse is obtained just by flipping the ordered pairs), and you can compose relations with `*`. Note that in general `r * r.inv` is not the identity; for this reason many people prefer to call it the "transpose" or "converse". You can also take the complement, union, and intersection of two relations that have the same type, and compare relations with `<=` and its friends. These all work like the same operators on the sets of ordered pairs. The `pToR` method converts a predicate on pairs to a relation.

Examples:

The relation < on `Int`. Its domain and range are `Int`, and its inverse is >.

The relation `r` given by the set of ordered pairs `s = {("a", 1), ("b", 2), ("a", 3)}`; `r = s.pred.pToR`; that is, turn the set into a predicate on ordered pairs and the predicate into a relation. Its inverse `r.inv = {(1, "a"), (2, "b"), (3, "a")}`, which is the sequence `{"a", "b", "a"}`. Its domain `r.dom = {"a", "b"}`; its range `r.rng = {1, 2, 3}`.

The advantage of relations is simplicity and generality; for example, there's no notion of "undefined" for relations. The drawback is that you can't write `r(x)` (although you can write `{x} * r` for the set of values related to `x` by `r`; see below).

A relation `r` has methods

`r.setF` to turn it into a set function: `r.setF(x)` is the set of elements that `r` relates to `x`. This is total. `Int."<".setF = (\i | {j: Int | j < i})`, and in the second example, `r.setF` maps `"a"` to `{1, 4}` and `"b"` to `{2}`. The inverse of `setF` is the `setRel` method for a function whose values are sets: `r.setF.setRel = r`, and `f.setRel.setF = f` if `f` yields sets.

`r.fun` to turn it into a function: `r.fun(x)` is undefined unless `r` relates `x` to exactly one value. Thus `r.fun = r.setF.one`.

If `s` is a set, `s.id` relates every member of the set to itself, and `s.rel` is a relation that relates `true` to each member of the set; thus it is `s.pred.inv.restrict({true})`. The relation's `rng` method inverts this: `s.rel.rng = s`.

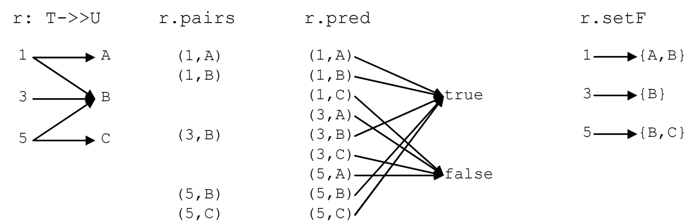
Viewing a set as a relation, you can compose it with a relation (or a function viewed as a relation); the result is the image of the set under the relation: `s * r = (s.rel * r).rng`. Note that this is never undefined, unlike sequence composition.

A relation `r: T->>U` can be viewed as a set `r.pairs` of pairs `(T,U)`, or as a total function `r.pred` on `(T,U)` that is `true` on the pairs that are in the relation, or as a function `r.setF` from `T` to `SET U`.

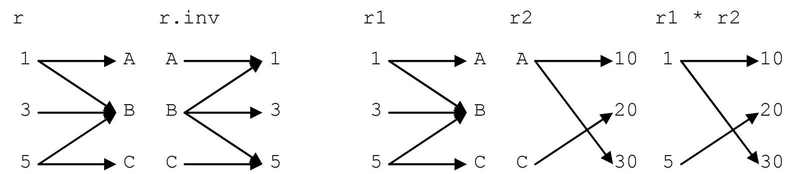
A method `m` of `U` is lifted to `SET U` and to relations to `U` just as it is to functions to `U` (see below), so that `r.m = r * U.m.rel`, as long as the set or relation doesn't have a `m` method.

The Boolean, set, and relational operators are extended to relations, so that `r1 \/ r2` is the union of the relations, `r1 /\ r2` the intersection, and `~r` the complement, and `r1 <= r2` iff `r1.pairs <= r2.pairs`.

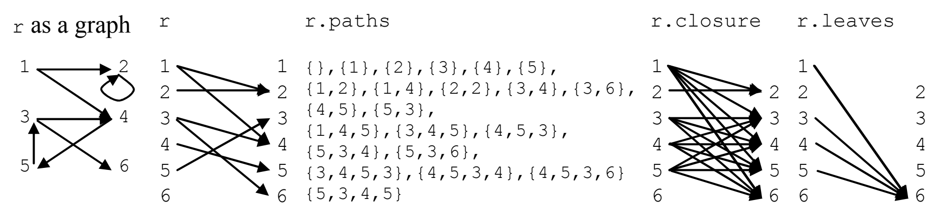
`T = {1,2,3,4,5}; U = {A,a,B,b,C}`



You can compute the inverse of a relation, and compose two relations by matching up the range of the first with the domain of the second.



If a relation `T->>T` has the same range and domain types it represents a graph, on which it makes sense to define the paths through the graph, and the transitive closure of the relation.



The partial inverse of `paths` is `pRel`; it takes a sequence to the relation that holds exactly between adjacent elements. So `\/ :r.paths.pRel = r`, and if the elements of `q` are distinct, `q` is the longest element of `q.pRel.paths`. The set `r.reachable(s)` is the elements reachable through `r` from a starting set `s`.

Method call	*Result type*	*Definition*
`r.pred`	`(T,U) ->Bool`	definition; `(\t,u | u IN r.setF(r))`
`r.pairs`	`SET (T,U)`	`{true} * r.pred.inv`
`r.set`	`SET T`	`r.rng`; only for `R = Bool->>T`
`r * rr`	`T->>V`	`(\t,v | (EXISTS u | r.pred(t,u) /\ rr.pred(u,v))).pToR` where `rr: U->>V`; works for `f` as well as `rr`
`r.dom`	`SET T`	`U.all * r.inv`
`r.rng`	`SET U`	`T.all * r`
`r.inv`	`U->>T`	`(\t,u | r.pred(u,t)).pToR`
`r.restrict(s)`	`T->>U`	`s.id * r` where `s: SET T`
`r.setF`	`T-> SET U`	`(\t | {t} * r)`
`r.fun`	`T->U`	`r.setF.one` (`one` is lifted from `SET U` to `T->SET U`)
`r.paths`	`SET SEQ T`	`{q:SEQ T | (ALL i IN q.dom-{0}|r.pred(q(i-1),q(i))) /\ ( q.rng.size=q.size \/ (q.head=q.last /\ q.rng.size=q.size-1))}` only for `R=T->>T`; paths self-intersect only at endpoints. See `q.pRel` for inverse.
`r.closure`	`T->>T`	`{q IN r.paths | q.size>1 || (q.head, q.last)}.pred.pToR` only for `R=T->>T`; there's a non-trivial path from `t1` to `t2`
`r.leaves`	`T->>T`	`r * (r.rng - r.dom).id` only for `R=T->>T`; there's a direct path from `t1` to `t2`, but nothing beyond.
`r.reachable(s)`	`SET T`	`(+ :{q :IN r.paths - {} | q.head IN s}).set`

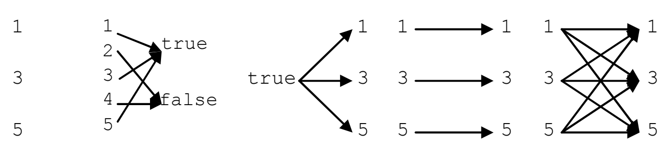
Sets

A set has methods for

> computing union, intersection, and set difference (lifted from `Bool`; see note 3 in section 4), and adding or removing an element, testing for membership and subset;
>
> choosing (deterministically) a single element from a set, or a sequence with the same members, or a maximum or minimum element, and turning a set into its characteristic predicate (the inverse is the predicate's `set` method);
>
> composing a set with a function or relation, and converting a set into a relation from `nil` to the members of the set (the inverse of this is just the range of the relation).

A set `s: SET T` can be viewed as a total function `s.pred` on `T` that is `true` on the members of `s` (sometimes called the 'characteristic function'), or as a relation `s.rel` from `true` to the members of the set, or as the identity relation `s.id` that relates each member to itself, or as the universal relation `s.univ` that relates all the members to each other.

`s ={1,3,5}` `s.pred` `s.rel = s.pred.inv .restrict ({true})` `s.id` `s.univ` `s.include`



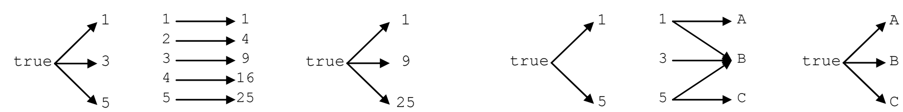
Error! Objects cannot be created from editing field codes.

You can compose a set `s` with a function or a relation to get another set, which is the image of `s` under the function or relation.

`{1,3,5} * square =` `{1,5} * r =`

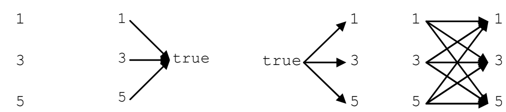
This is just like relational composition on `s.rel`.

`{1,3,5}.rel * square =` `{1,5}.rel * r =`



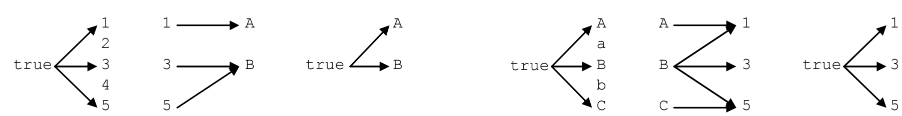
The universal relation `s.univ` is just the composition of `s.rel` with its inverse:

`s ={1,3,5}` `s.rel.inv` `* s.rel =` `s.univ`



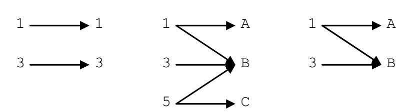
You can compute the range and domain of a relation. An element `t` is in the range if `r` relates something to it, and in the domain if `r` relates it to something. (For clarity, the figures show the relations corresponding to the sets, not the sets themselves.)

`T.all *` `r =` `r.rng` `U.all *` `r.inv =` `r.dom`



You can restrict the domain of a relation or function to a set `s` by composing the identity relation `s.id` with it. To restrict the range to `s`, use the same idea and write `r * s.id`.

`{1,3}.id *` `r =` `r.restrict( {1,3})`



You can convert a set of pairs `s` to a relation with `s.pred.pToR`; there are examples in the section on relations above.

You can pick out one element of a set s with `s.choose`. This is deterministic: `choose` always returns the same value given the same set (a necessary property for it to be a function). It is undefined if the set is empty. A variation of `choose` is `one`: `s.one` is undefined unless `s` has exactly one element, in which case it returns that element.

You can compute the set of all permutations of a set; a permutation is a sequence, explained below. You can sort a set or compute its maximum or minimum; note that the results make an arbitrary choice if the ordering function is not a total order. You can also compute the "leaves" that a relation computes from a set: the extremal points where the relation takes the elements of the set; here you get them all, so there's no need for an arbitrary choice. If you think of the graph induced by the closure of the relation, starting from the elements of the set, then the leaves are the nodes of the graph that have no outgoing edges (successors).

```
s = {3,1,5}, s.perms ={{3,1,5},{3,5,1},{5,1,3},{5,3,1},{{1,3,5},{1,5,3}},
s.sort = {1,3,5}, s.max = 5, s.min = 3.
```

<table>
<tr><th>Method call</th><th>Result type</th><th>Definition</th></tr>
<tr><td>s.pred</td><td>T->Bool</td><td>definition; (\t | t IN s)</td></tr>
<tr><td>s.rel</td><td>Bool->>T</td><td>s.pred.inv</td></tr>
<tr><td>s.id</td><td>T->>T</td><td>(\ t1,t2 | t1 IN s /\ t1 = t2)</td></tr>
<tr><td>s.univ</td><td>T->>T</td><td>s.rel.inv * s.rel</td></tr>
<tr><td>s.include</td><td>SET T->>T</td><td>(\ st: SET T, t | t IN (st /\ s)).pToR</td></tr>
<tr><td>t IN s</td><td>Bool</td><td>s.pred(t)</td></tr>
<tr><td>s1 <= s2</td><td>Bool</td><td>s1 /\ s2 = s1, or equivalently (∀ t | t IN s1 ==> t IN s2)</td></tr>
<tr><td>s1 /\ s2</td><td>S</td><td>(\t | t IN s1 /\ t IN s2) intersection</td></tr>
<tr><td>s1 \/ s2</td><td>S</td><td>(\t | t IN s1 \/ t IN s2) union</td></tr>
<tr><td>~ s</td><td>S</td><td>(\t | ~(t IN s))</td></tr>
<tr><td>s1 - s2</td><td>S</td><td>s1 /\ ~ s2</td></tr>
<tr><td>s * r</td><td>SET U</td><td>(s.rel * r).rng where R=T->>U; works for f as well as r</td></tr>
<tr><td>s.size</td><td>Nat</td><td>s.seq.dom.max + 1</td></tr>
<tr><td>s.choose</td><td>T</td><td>?</td></tr>
<tr><td>s.one</td><td>T</td><td>(s.size = 1 => s.choose); undefined if s#{t}</td></tr>
<tr><td>s.perms</td><td>SET Q</td><td>{q: SEQ T | q.size = s.size /\ q.rng = s}</td></tr>
<tr><td>s.seq</td><td>Q</td><td>s.perms.choose</td></tr>
<tr><td>s.fsort(f)</td><td>Q</td><td>{q IN s.perms|(∀i IN q.dom-{0}|f(q(i),q(i-1)))}.choose</td></tr>
<tr><td>s.sort</td><td>Q</td><td>s.fsort(T."<=")</td></tr>
<tr><td>s.fmax(f)</td><td>T</td><td>s.fsort(f).last and likewise for fmin</td></tr>
<tr><td>s.max</td><td>T</td><td>s.sort.last and likewise for min. Note that this is not the same as /\ : s, unless s is totally ordered.</td></tr>
<tr><td>s.leaves(r)</td><td>S</td><td>r.restrict(s).closure.leaves.rng; generalizes max</td></tr>
<tr><td>s.combine(f)</td><td>T</td><td>s.seq.combine(f); useful if f is commutative</td></tr>
</table>

Functions

A function is a set of ordered pairs; the first element of each pair comes from the function's *domain*, and the second from its *range*. A function produces at most one value for an argument; that is, two pairs can't have the same first element. Thus a function is a relation in which each element of the domain is related to at most one thing. A function may be partial, that is, undefined at some elements of its domain. The expression f!x is true if f is defined at x, false otherwise. Like everything (except types), functions are ordinary values in Spec.

Given a function, you can use a function constructor to make another one that is the same except at a particular argument, as in the DB example in the section on constructors below. Another example is f{x -> 0}, which is the same as f except that it is 0 at x. If you have never seen a construction like this one, think about it for a minute. Suppose you had to implement it. If f is represented as a table of (argument, result) pairs, the code will be easy. If f is represented by code that computes the result, the code for the constructor is less obvious, but you can make a new piece of code that says

```
(\ y: Int | ( (y = x) => 0 [*] f(y) ))
```

Here '\' is 'lambda', and the subexpression ((y = x) => 0 [*] f(y)) is a conditional, modeled on the conditional commands we saw in the first section; its value is 0 if y = x and f(y) otherwise, so we have changed f just at 0, as desired. If the else clause [*] f(y) is omitted, the condition is undefined if y # x. Of course in a running program you probably wouldn't want to construct new functions very often, so a piece of Spec that is intended to be close to practical code must use function constructors carefully.

Functions can return functions as results. Thus T->U->V is the type of a function that takes a T and returns a function of type U->V, which in turn takes a U and returns a V. If f has this type, then f(t) has type U->V, and f(t)(u) has type V. Compare this with (T, U)->V, the type of a function which takes a T and a U and returns a V. If g has this type, g(t) doesn't type-check, and g(t, u) has type V. Obviously f and g are closely related, but they are not the same. Functions declared with more than one argument are a bit tricky; they are discussed in the section on tuples below.

You can define your own functions either by lambda expressions like the one above, or more generally by function declarations like this one

```
FUNC NewF(y: Int) -> Int = RET ( (y = x) => 0 [*] f(y) )
```

The value of this NewF is the same as the value of the lambda expression. To avoid some redundancy in the language, the meaning of the function is defined by a command in which RET subcommands specify the value of the function. The command might be syntactically non-deterministic (for instance, it might contain VAR or []), but it must specify at most one result value for any argument value; if it specifies no result values for an argument or more than one value, the function is undefined there. If you need a full-blown command in a function constructor, you can write it with LAMBDA instead of \:

```
(LAMBDA (y: Int) -> Int = RET ( (y = x) => 0 [*] f(y) ))
```

You can *compose* two functions with the * operator, writing f * g. This means to apply f first and then g, so you read it "f then g". It is often useful when f is a sequence (remember that a SEQ T is a function from {0, 1, ..., size-1} to T), since the result is a sequence with every element of f mapped by g. This is Lisp's or Scheme's "map". So:

```
(0 .. 4) * {\ i: Int | i*i} = (SEQ Int){0, 1, 4, 9, 16}
```

since 0 .. 4 = {0, 1, 2, 3, 4} because Int has a method .. with the obvious meaning: i .. j = {i, i+1, ..., j-1, j}. In the section on constructors below we see another way to write

```
(0 .. 4) * {\ i: Int | i*i},
```

as

```
{i :IN 0 .. 4 || i*i}.
```

This is more convenient when the mapping function is defined by an expression, as it is here, but it's less convenient if the mapping function already has a name. Then it's shorter and clearer to write

```
(0 .. 4) * factorial
```

rather than

```
{i :IN 0 .. 4 || factorial(i)}.
```

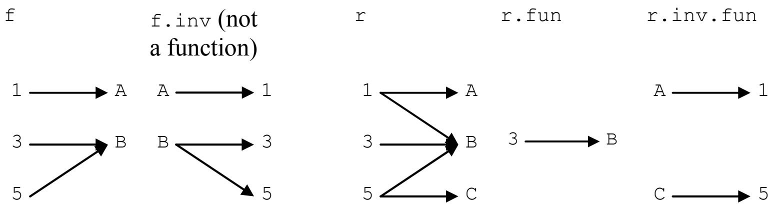
A function f has methods f.dom and f.rng that yield its domain and range sets, f.inv that yields its inverse (which is undefined at y unless f maps exactly one argument to y), and f.rel that turns it into a relation (see below). f.restrict(s) is the same as f on elements of s and undefined elsewhere. The *overlay* operator combines two functions, giving preference to the second: (f1 + f2)(x) is f2(x) if that is defined and f1(x) otherwise. So f{3 -> 24} = f + {3 -> 24}.

If type `U` has method `m`, then the function type `F = T->U` has a "lifted" method `m` that composes `U.m` with `f`, unless `F` already has a `m` method. `F. m` is defined by

```
(\ f | (\ t | f(t).m))
```

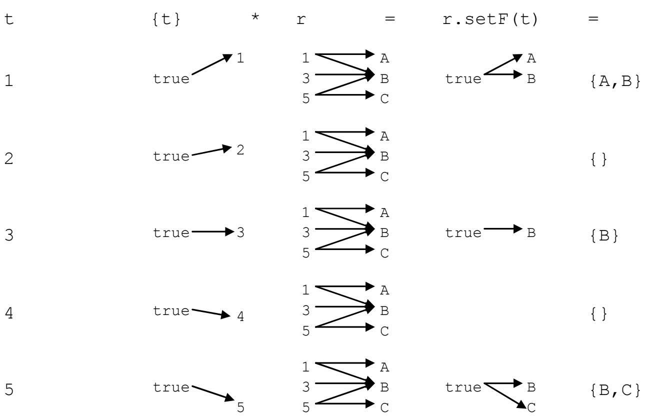
so that `f.m = f * U.m`. For example, `{1, 3, 5}.square = {1, 9, 25}`. If `m` takes a second argument of type `W`, then `F.m` takes a second argument of the same type and uses it uniformly. This also works for sets and relations.

You can turn a relation into a function by discarding all the pairs whose first element is related to more than one thing

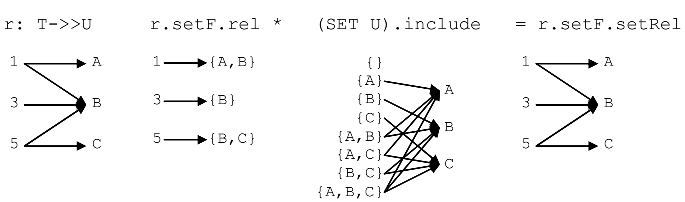


You can go back and forth between a relation `T->>U` and a function `T->SET U` with the `setF` and `setRel` methods.

```
r.setF = (\t | {t} * r)
```




```
f.setRel = f.rel.include
```



<table>
<tr><th>Method call
f has type T->U</th><th>Result type</th><th>Definition</th></tr>
<tr><td>f + f'</td><td>T->U</td><td>(f.rel \/ (f'.rel * f1.rng.id)).func
(\t | (f!t => f (t) [*] f'(t)))</td></tr>
<tr><td>f!t</td><td>Bool</td><td>t IN f.dom</td></tr>
<tr><td>f!!t</td><td>Bool</td><td></td></tr>
<tr><td>f $ t</td><td>U</td><td>Applies f to the tuple t; see the section on records below</td></tr>
<tr><td>f * g</td><td>T->V</td><td>(f.rel * g.rel).fun, where g:U->V</td></tr>
<tr><td>f.rel</td><td>T->>U</td><td>(\t,u | f!t /\ f(t) = u). pToR</td></tr>
<tr><td>f.setRel</td><td>T->>V</td><td>f.rel.include, only for F=T->SET V</td></tr>
<tr><td>f.set</td><td>SET T</td><td>f.restrict({true}).rng, only for F=T->Bool</td></tr>
<tr><td>f.pToR</td><td>V->>W</td><td>definition, only for F=(V,W)->Bool; (\v |{w|f(v,w)}).setRel</td></tr>
</table>

A function type `F = T->U` also has a set of *lifting* methods that turn an `f` into a function on `SET T`, `V->T`, or `V->>T` by composition. This works for `F = (T,W)->U` as well; the lifted method also takes a `W` and uses it uniformly. A relation type `R = T->>U` is also lifted to `SET T`. These are used to automatically supply the higher-order types with lifted methods.

<table>
<tr><th>Method</th><th>method m of type T, with type F</th><th>makes method m for type</th><th>with type</th><th>by</th></tr>
<tr><td>f.liftSet</td><td>T-> U</td><td>S =SET T</td><td>SET T -> SET U</td><td>s .m=(s * f).set</td></tr>
<tr><td>f.liftFun</td><td>T-> U</td><td>FF=V-> T</td><td>(V-> T)->(V-> U)</td><td>ff.m=ff * f</td></tr>
<tr><td>f.liftRel</td><td>T-> U</td><td>RR=V->>T</td><td>(V->>T)->(V->>U)</td><td>ff.m=rr * f</td></tr>
<tr><td>f.liftSet</td><td>(T,W)->U</td><td>S =SET T</td><td>(SET T ,W)->SET U</td><td>s .m(w)=(s *(\t|f(t,w)).set</td></tr>
<tr><td>f.liftFun</td><td>(T,W)->U</td><td>FF=V-> T</td><td>((V-> T),W)->(V-> U)</td><td>ff.m(w)=ff *(\t|f(t,w))</td></tr>
<tr><td>f.liftRel</td><td>(T,W)->U</td><td>RR=V->>T</td><td>((V->>T),W)->(V->>U)</td><td>ff.m(w)=rr *(\t|f(t,w))</td></tr>
<tr><td></td><td>with type R</td><td></td><td></td><td></td></tr>
<tr><td>r.liftSet</td><td>T->>U</td><td>S =SET T</td><td>SET T -> SET U</td><td>s .m=(s * r).set</td></tr>
</table>

Changing coordinates: relations, predicates, sets, functions, and sequences

As we have seen, there are several ways to view a set or a relation. Which one is best depends on what you want to do with it, and what is familiar and comfortable in your application. Often the choice of representation makes a big difference to the convenience and clarity of your code, just as the choice of coordinate system makes a big difference in a physics problem. The following tables summarize the different representations, the methods they have, and the conversions among them. The players are sets, functions, predicates, and relations.

<table>
<tr><th>Method</th><th>Converts</th><th>to</th><th>by</th><th>Inverse</th></tr>
<tr><td>.rel</td><td>F=T->U</td><td>T->>U</td><td>(\t,u|f!t/\f(t)=u).pToR</td><td>.fun</td></tr>
<tr><td></td><td>S=SET T</td><td>Bool->>T</td><td>s.pred.inv.restrict({true})</td><td>.set</td></tr>
<tr><td>.pred</td><td>S=SET T</td><td>T->Bool</td><td>definition; (\t | t IN s)</td><td>.set</td></tr>
<tr><td></td><td>R=T->>U</td><td>(T,U)->Bool</td><td>definition;
(\t,u | u IN r.setF(r))</td><td>.pToR</td></tr>
<tr><td>.set</td><td>F=T->Bool</td><td>SET T</td><td>f.restrict({true}).rng</td><td>.rel</td></tr>
<tr><td></td><td>R=Bool->>T</td><td>SET T</td><td>r.rng</td><td>.rel</td></tr>
<tr><td>.fun</td><td>R=T->>U</td><td>T->U</td><td>r.setF.one</td><td>.rel</td></tr>
<tr><td>.pToR</td><td>F=(T,U)->Bool</td><td>T->>U</td><td>definition;
(\t | {u|f(t,u)}.setRel</td><td>.pred</td></tr>
<tr><td>.setF</td><td>R=T->>U</td><td>T->SET U</td><td>(\t | {t} * r)</td><td>.setRel</td></tr>
<tr><td>.setRel</td><td>F=T->SET U</td><td>T->>U</td><td>f.rel.include</td><td>.setF</td></tr>
<tr><td>.paths</td><td>T->>T</td><td>SET SEQ T</td><td>see above</td><td>.pRel</td></tr>
<tr><td>.pRel</td><td>SEQ T</td><td>T->>T</td><td>{i :IN q.dom - {0} ||
(q(i-1), q(i))}.pred.pToR</td><td>.paths
sort of</td></tr>
</table>

Type	*based on convert with*	*equivalent to convert with*	*converts to with*	*methods*
SET T	T->Bool .pred		Bool->>T .rel	IN <= /\ \/ ~ - * pred rel id size choose one perms seq sort max
T->Bool		SET T .set		function methods and .set
Bool->>T		SET T .set		relation methods and .set
T->>U	(T,U)->Bool .pred	SET (T,U) .pred.set T->SET U .setF	T->U .fun	* dom rng inv restrict setF fun [paths closure]
(T,U)->Bool		T->>U .pToR SET (T,U) *.set*		function methods and .pToR .set
T->SET U		T->>U .setRel		function methods and .setRel
SET (T,U)	(T,U)->Bool *.pred*	T->>U .pred.pToR		just a set
T->U			T->>U .rel	* dom rng inv restrict (from T->>U), + ! !! rel
SEQ T	Int->T			function methods and + <= <<= size seg sub head tail addh remh last reml addl fill tuple lexLE

Here is another way to look at it. Each of the types that label rows and columns in the following tables is equivalent to the others, and the entries in the table tell how to convert from one form to another.

from	**to**	*set* SET T	*predicate* T->Bool	*relation* Bool->>T
set	SET T		.pred	.rel
predicate	T->Bool	.set		.inv
relation	Bool->>T	.set	.inv	

from	**to**	*relation* T->>U	*predicate* (T,U)->Bool	*set function* T->SET U	*set of pairs* SET (T,U)
relation	T->>U		.pred	.setF	.pred.set
predicate	(T,U)->Bool	.pToR		.pToR.setF	.set
set function	T->SET U	.setRel	.setRel.pred		.setRel.pred.set
set of pairs	SET (T,U)	.pred.pToR	.pred	.pred.pToR.setF	

Sequences

A function is called a sequence if its domain is a finite set of consecutive `Int`'s starting at 0, that is, if it has type

```
Q = Int->T SUCHTHAT q.dom = {i: Int | 0<=i /\ i<q.dom.max }
```

We denote this type (with the methods defined below) by `SEQ T`. A sequence inherits the methods of the function (though it overrides `+`), and it also has methods for

detaching or attaching the first or last element,
extracting a segment of a sequence, concatenating two sequences, or finding the size,

making a sequence with all elements the same: `t.Fill(n)`,
testing for prefix or sub-sequence (not necessarily contiguous): `q1<=q2`, `q1<<-q2`,
lexical comparison, permuting, and sorting,
filtering, iterating over, and combining the elements,
making a sequence into a relation that holds exactly between successive elements,
treating a sequence as a multiset with operations to:
count the number of times an element appears: `q.count(t)`,
test membership: `t IN q`,
take differences: `q1 - q2`
(`"+"` is union and `addl` adds an element; to remove an element use `q - {t}`; to test equality use `q1 IN q2.perms`).

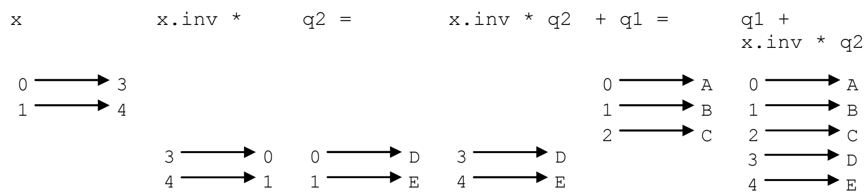
All these operations are undefined if they use out-of-range subscripts, except that a sub-sequence is always defined regardless of the subscripts, by taking the largest number of elements allowed by the size of the sequence.

The value of `i .. j` is the sequence of integers from `i` to `j`.

To apply a function `f` to each of the elements of `q`, just use composition `q * f`.

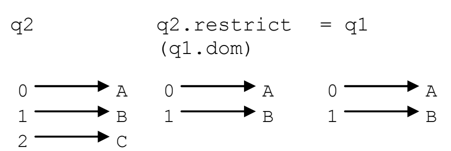
The `"+"` operator concatenates two sequences.

`q1 + q2 = q1 + x.inv * q2`, where `x = (q1.size .. q1.size+q2.size-1)`
`q1 = {A,B,C}; q2 = {D,E}; x = {3,4}; q1 + q2 = {A,B,C,D,E}`



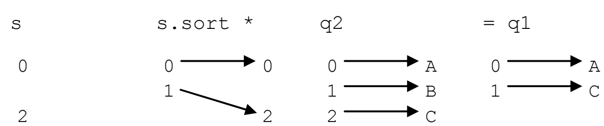
You can test for `q1` being a prefix of `q1` with `q1 <= q2`, and for it being an arbitrary subsequence, not necessarily contiguous, with `q1 <<= q2`.

`q1 <= q2 = (q1 = q2.restrict(q1.dom))`
`q1 = {A,B}; q2 = {A,B,C}`




```
q1 <<= q2 = (EXISTS s: SET Int | s <= q2.dom /\ q1 = s.sort * q2
```

`q1 = {A,C}; q2 = {A,B,C}`; choose `s = {0,2} <= {0,1,2}`



You can take a subsequence of size `n` starting at `i` with `q.seg(i,n)` and a subsequence from `i1` to `i2` with `q.sub(i1,i2)`.

```
q.seg(i,n) = (i .. i+n-1) * q
q = {A,B,C}; i = 1; n = 3; q.seg(1,3) = {B,C}

i .. i+n-1  * q           =q.seg(i,n)

0 ──► 1    0 ──► A     0 ──► B
1 ──► 2    1 ──► B     1 ──► C
2 ──► 3    2 ──► C
```

You can select the elements of `q` that satisfy a predicate `f` with `q.filter(f)`.

```
q.filter(f) = (q * f).set.sort * q
q = {5,4,3,2,1}; f = even

q          q * f        (q * f).set   .sort       * q =

0 ──► 5    0 ──► false                0 ──► 1     0 ──► 4
1 ──► 4    1 ──► true     1           1 ──► 3     1 ──► 2
2 ──► 3    2 ──► false
3 ──► 2    3 ──► true     3
4 ──► 1    4 ──► false
```

You can zip up a pair of sequences to get a sequence of pairs with `q1 || q2`. Then you can compose a binary function to get the sequence of results

```
q1={1,2,3,4,5}  q2={6,7,8,9,10}  q1 || q2         (q1 || q2) * Int."+"

0 ──► 1         0 ──► 6          0 ──► (1,6)      0 ──► 7
1 ──► 2         1 ──► 7          1 ──► (2,7)      1 ──► 9
2 ──► 3         2 ──► 8          2 ──► (3,8)      2 ──► 11
3 ──► 4         3 ──► 9          3 ──► (4,9)      3 ──► 13
4 ──► 5         4 ──► 10         4 ──► (5,10)     4 ──► 15
```

Since a pair of `SEQ T` is a function `0..1 -> 0..n -> T` and `SEQ (T, T)` is a function `0..n -> 0..1 -> T`, `zip` just reverses the order of the arguments.

You can apply a combining function `f` successively to the elements of `q` with `q.iterate(f)`. To get the result of combining all the elements of `q` with `f` use `q.combine(f) = q.iterate(f).last`. The syntax `+ : q` is short for `q.combine(T."+")`; it works for any binary operator that yields a `T`.

```
q = {1,2,3,4,5}  q.iterate(Int."+")

0 ──► 1          0 ──► 1
1 ──► 2          1 ──► 3
2 ──► 3          2 ──► 6
3 ──► 4          3 ──► 10
4 ──► 5          4 ──► 15
```



```
Method call      Result     Definition
                 type
q1 + q2          Q          q1 + (q1.size .. q1.size+q2.size-1).inv * q2
q1 <= q2         Bool       q1 = q2.restrict(q1.dom)
q1 <<= q2        Bool       (EXISTS s: SET Int | s <= q2.dom /\ q1 = s.sort * q2)
q.size           Nat        q.dom.size
q.seg(i,n)       Q          (i .. i+n-1) * q
q.sub(i1,i2)     Q          (i1 .. i2) * q
q.head           T          q(0)
q.tail           Q          (q # {} => q.sub(1, q.size-1))
t.fill(n)        Q          (0 .. n-1) * {* -> t}
q1.lexLE(q2,f)   Bool       (EXISTS q,n | n=q.size /\ q<=q1 /\ q<=q2 /\
                                 (q=q1 \/ f(q1(n),q2(n)) /\ q1(n)#q2(n)))
q.filter(f)      Q          (q * f).set.sort * q, where f: T->Bool
q || qU          SEQ(T,U)   RET (\ i | (i IN (q.dom /\ qU.dom) => (q(i), qU(i))))
                            where qU: SEQ U
q.iterate(f)     Q          {qr |   qr.size=q.size /\ qr(0)=q(0)
                                 /\ (ALL i IN q.dom-{0}|qr(i)=f(qr(i-1),q(i)))}.one
                            where f: (T,T)->T
q.combine(f)     T          q.iterate.last
t ** n           T          t.fill(n).combine(T."*")
q.pRel           T->>T      {i :IN q.dom - {0} || (q(i-1), q(i))}.pred.pToR
q.count(t)       Nat        {t' :IN q | t' = t}.size
t IN q           Bool       t IN q.rng
q1 - q2          Q          {q |(ALL t | q.count(t)={q1.count(t)-q2.count(t), 0}.max)}.
                            choose
```

`SEQ T` has the same `perms`, `fsort`, `sort`, `fmax`, `fmin`, `max`, and `min` constructors as `SET T`.

Records and tuples

Sets, functions, and sequences are good when you have many values of the same type. When you have values of different types, you need a tuple or a record (they are the same, except that a record allows you to name the different values). In Spec a record is a function from the string names of its fields to the field values, and an *n*-tuple is a function from `0..n-1` to the field values. There is special syntax for declaring records and tuples, and for reading and writing record fields:

`[f: T, g: U]` declares a record with fields `f` and `g` of types `T` and `U`.

`(T, U)` declares a tuple with fields of types `T` and `U`.

`r.f` is short for `r("f")`, and `r.f := e` is short for `r := r{"f"->e}`.

There is also special syntax for constructing record and tuple values, illustrated in the following example. Given the type declaration

```
TYPE Entry = [salary: Int, birthdate: String]
```

we can write a record value

```
Entry{salary := 23000, birthdate := "January 3, 1955"}
```

which is short for the function constructor

```
Entry{"salary" -> 23000, "birthdate" -> "January 3, 1955"}.
```

The constructor `(23000, "January 3, 1955")` yields a tuple of type `(Int, String)`. It is short for

```
{0 -> 23000, 1 -> "January 3, 1955"}
```

This doesn't work for a singleton tuple, since `(x)` has the same value as `x`. However, the sequence constructor `{x}` will do for constructing a singleton tuple, since a singleton `SEQ T` is also a singleton tuple; in fact, this is the only way to write the type of a singleton tuple, since `(T)` is the same as `T` because parentheses are used for grouping in types just as they are in ordinary expressions.

The type of a record is `String->Any SUCHTHAT ...`, and the type of a tuple is `Nat->Any SUCHTHAT ...`. Here the `SUCHTHAT` clauses are of the form `this("f") IS T`; they specify the types of the fields. In addition, a record type has a method called `fields` whose value is the sequence of field names (it's the same for every record). Thus `[f: T, g: U]` is short for

```
String->Any WITH { fields:=(\r: String->Any | (SEQ String){"f", "g"}) }
            SUCHTHAT   this.dom >= {"f", "g"}
                    /\ this("f") IS T /\ this("g") IS U
```

A tuple type works the same way; its `fields` is just `0..n-1` if the tuple has `n` fields. Thus `(T, U)` is short for

```
Int->Any WITH { fields:=(\r: Int->Any | 0..1) }
            SUCHTHAT   this.dom = 0..1
                    /\ this(0) IS T /\ this(1) IS U
```

Thus to convert a record `r` into a tuple, write `r.fields * r`, and to convert a tuple `t` into a record, write `r.fields.inv * t`.

There is no special syntax for tuple fields, since you can just write `t(2)` and `t(2) := e` to read and write the third field, for example (remember that fields are numbered from 0).

Functions declared with more than one argument are a bit tricky: they take a single argument that is a tuple. So `f(x: Int)` takes an `Int`, but `f(x: Int, y: Int)` takes a tuple of type `(Int, Int)`. This convention keeps the tuples in the background as much as possible. The normal syntax for calling a function is `f(x, y)`, which constructs the tuple `(x, y)` and passes it to `f`. However, `f(x)` is treated differently, since it passes `x` to `f`, rather than the singleton tuple `{x}`. If you have a tuple `t` in hand, you can pass it to `f` by writing `f$t` without having to worry about the singleton case; if `f` takes only one argument, then `t` must be a singleton tuple and `f$t` will pass `t(0)` to `f`. Thus `f$(x, y)` is the same as `f(x, y)` and `f${x}` is the same as `f(x)`.

A function declared with names for the arguments, such as

```
(\ i: Int, s: String | i + StringToInt(x))
```

has a type that ignores the names, `(Int, String)->Int`. However, it also has a method `argNames` that returns the sequence of argument names, `{"i", "s"}` in the example, just like a record. This makes it possible to match up arguments by name, as in the following example.

A database is a set `s` of records. A selection query `q` is a predicate that we want to apply to the records. How do we get from the field names, which are strings, to the argument for `q`? Assume that `q` has an `argNames` method. So if `r IN s`, `q.argNames * r` is the tuple that we want to feed to `q`; `q$(q.argNames * r)` is the query, where `$` is the operator that applies a function to a tuple of its arguments.

There is one problem if not all fields are defined in all records: when we try to use `q.argNames * r`, it will be undefined if `r` doesn't have all the fields that `q` wants. We want to apply it only to the records in `s` that have all the necessary fields. That is the set

```
{r :IN s | q.argNames <= r.fields}
```

The answer we want is the subset of records in this set for which `q` is true. That is

```
{r :IN s | q.argNames <= r.fields /\ q$(q.argNames * r)}
```


To project the database, discarding all the fields except the ones in `projection` (a set of strings), write

```
{r :IN s || r.restrict(projection)}
```

Constructors

Functions, sets, and sequences make it easy to toss large values around, and constructors are special syntax to make it easier to define these values. For instance, you can describe a database as a function `db` from names to data records with two fields:

```
TYPE DB = (String -> Entry)
TYPE Entry = [salary: Int, birthdate: Int]
VAR db := DB{}
```

Here `db` is initialized using a function constructor whose value is a function undefined everywhere. The type can be omitted in a variable declaration when the variable is initialized; it is taken to be the type of the initializing expression. The type can also be omitted when it is the upper case version of the variable name, `DB` in this example.

Now you can make an entry with

```
db := db{ "Smith" -> Entry{salary := 23000, birthdate := 1955} }
```

using another function constructor. The value of the constructor is a function that is the same as `db` except at the argument `"Smith"`, where it has the value `Entry{...}`, which is a record constructor. This assignment could also be written

```
db("Smith") := Entry{salary := 23000, birthdate := 1955}
```

which changes the value of the `db` function at `"Smith"` without changing it anywhere else. This is actually a shorthand for the previous assignment. You can omit the field names if you like, so that

```
db("Smith") := Entry{23000, 1955}
```

has the same meaning as the previous assignment. Obviously this shorthand is less readable and more error-prone, so use it with discretion. Another way to write this assignment is

```
db("Smith").salary := 23000; db("Smith").birthdate := 1955
```

A record is actually a function as well, from the strings that represent the field names to the field values. Thus `Entry{salary := 23000, birthdate := 1955}` is a function `r: String->Any` defined at two string values, `"salary"` and `"birthdate"`: `r("salary") = 23000` and `r("birthdate") = 1955`. We could have written it as a function constructor `Entry{"salary" -> 23000, "birthdate" -> 1955}`, and `r.salary` is just a convenient way of writing `r("salary")`.

The set of names in the database can be expressed by a set constructor. It is just

```
{n: String | db!n},
```

in other words, the set of all the strings for which the `db` function is defined ('`!`' is the 'is-defined' operator; that is, `f!x` is true iff `f` is defined at `x`). Read this "the set of strings `n` such that `db!n`". You can also write it as `db.dom`, the domain of `db`; section 9 of the reference manual defines lots of useful built in methods for functions, sets, and sequences. It's important to realize that you can freely use large (possibly infinite) values such as the `db` function. You are writing a spec, and you don't need to worry about whether the compiler is clever enough to turn an expensive-looking manipulation of a large object into a cheap incremental update. That's the implementer's problem (so you may have to worry about whether *she* is clever enough).

If we wanted the set of lengths of the names, we would write

```
{n: String | db!n || n.size}
```

This three part set constructor contains `i` if and only if there exists an `n` such that `db!n` and `i = n.size`. So `{n: String | db!n}` is short for `{n: String | db!n || n}`. You can introduce more than one name, in which case the third part defaults to the last name. For example, if we represent a directed graph by a function on pairs of nodes that returns `true` when there's an edge from the first to the second, then

```
{n1: Node, n2: Node | graph(n1, n2) || n2}
```

is the set of nodes that are the target of an edge, and the "`|| n2`" could be omitted. This is just the range `graph.rng` of the relation `graph` on nodes.

Following standard mathematical notation, you can also write

```
{f :IN openFiles | f.modified}
```

to get the set of all open, modified files. This is equivalent to

```
{f: File | f IN openFiles /\ f.modified}
```

because if `s` is a `SET T`, then `IN s` is a type whose values are the `T`'s in `s`; in fact, it's the type `T SUCHTHAT (\ t | t IN s)`. This form also works for sequences, where the second operand of `:IN` provides the ordering. So if `s` is a sequence of integers, `{x :IN s | x > 0}` is the positive ones, `{x :IN s | x > 0 || x * x}` is the squares of the positive ones, and `{x :IN s || x * x}` is the squares of all the integers, because an omitted predicate defaults to `true`.[5]

To get sequences that are more complicated you can use sequence generators with `BY` and `WHILE`. You can skip this paragraph until you need to do this.

```
{i := 1 BY i + 1 WHILE i <= 5 | true || i}
```

is `{1, 2, 3, 4, 5}`; the second and third parts could be omitted. This is just like the "for" construction in C. An omitted `WHILE` defaults to `true`, and an omitted `:=` defaults to an arbitrary choice for the initial value. If you write several generators, each variable gets a new value for each value produced, but the second and later variables are initialized first. So to get the sums of successive pairs of elements of `s`, write

```
{x := s BY x.tail WHILE x.size > 1 || x(0) + x(1)}
```

To get the sequence of partial sums of `s`, write (eliding `|| sum` at the end)

```
{x :IN s, sum := 0 BY sum + x}
```

Taking `last` of this would give the sum of the elements of `s`. To get a sequence whose elements are reversed from those of `s`, write

```
{x :IN s, rev := {} BY {x} + rev}.last
```

To get the sequence {e, f(e), f^2(e), ..., f^n(e)}, write

```
{i :IN 1 .. n, iter := e BY f(iter)}
```

Combinations

A combination is a way to combine the elements of a non-empty sequence or set into a single value using an infix operator, which must be associative, and must be commutative if it is applied to a set. You write "operator `:` sequence or set". This is short for `q.combine(T.operator)`. Thus

```
+ : (SEQ String){"He", "l", "lo"} = "He" + "l" + "lo" = "Hello"
```

because `+` on sequences is concatenation, and

```
+ : {i :IN 1 .. 4 || i**2} = 1 + 4 + 9 + 16 = 30
```

Existential and universal quantifiers make it easy to describe properties without explaining how to test for them in a practical way. For instance, a predicate that is `true` iff the sequence `s` is sorted is

```
(ALL i :IN 1 .. s.size-1   | s(i-1) <= s(i))
```

This is a common idiom; read it as

[5] In the sequence form, `IN s` is not a set type but a special construct; treating it as a set type would throw away the essential ordering information.

"for all `i` in `1 .. s.size-1`, `s(i-1) <= s(i)`".

This could also be written

```
(ALL i :IN (s.dom - {0}) | s(i-1) <= s(i))
```

since `s.dom` is the domain of the function `s`, which is the non-negative integers `< s.size`. Or it could be written

```
(ALL i :IN s.dom | i > 0 ==> s(i-1) <= s(i))
```

Because a universal quantification is just the conjunction of its predicate for all the values of the bound variables, it is simply a combination using `/\` as the operator:

```
(ALL    i | Predicate(i)) = /\ : {i | Predicate(i)}
```

Similarly, an existential quantification is just a similar disjunction, hence a combination using `\/` as the operator:

```
(EXISTS i | Predicate(i)) = \/ : {i | Predicate(i)}
```

Spec has the redundant `ALL` and `EXISTS` notations because they are familiar.

If you want to get your hands on a value that satisfies an existential quantifier, you can construct the set of such values and use the `choose` method to pick out one of them:

```
{i | Predicate(i)}.choose
```

The `VAR` command described in the next section on commands is another form of existential quantification that lets you get your hands on the value, but it is non-deterministic.

Commands

Commands are for changing the state. Spec has a few simple commands, and seven operators for combining commands into bigger ones. The main simple commands are assignment and routine invocation. There are also simple commands to raise an exception, to return a function result, and to `SKIP`, that is, do nothing. If a simple command evaluates an undefined expression, it fails (see below).

You can write `i + := 3` instead of `i := i + 3`, and similarly with any other binary operator.

The operators on commands are:

- A conditional operator: `predicate => command`, read "if `predicate` then `command`". The predicate is called a *guard*.
- Choice operators: `c1 [] c2` and `c1 [*] c2`, read 'or' and 'else'.
- Sequencing operators: `c1 ; c2` and `c1 EXCEPT handler`. The `handler` is a special form of conditional command: `exception => command`.
- Variable introduction: `VAR id: T | command`, read "choose `id` of type `T` such that `command` doesn't fail".
- Loops: `DO command OD`.

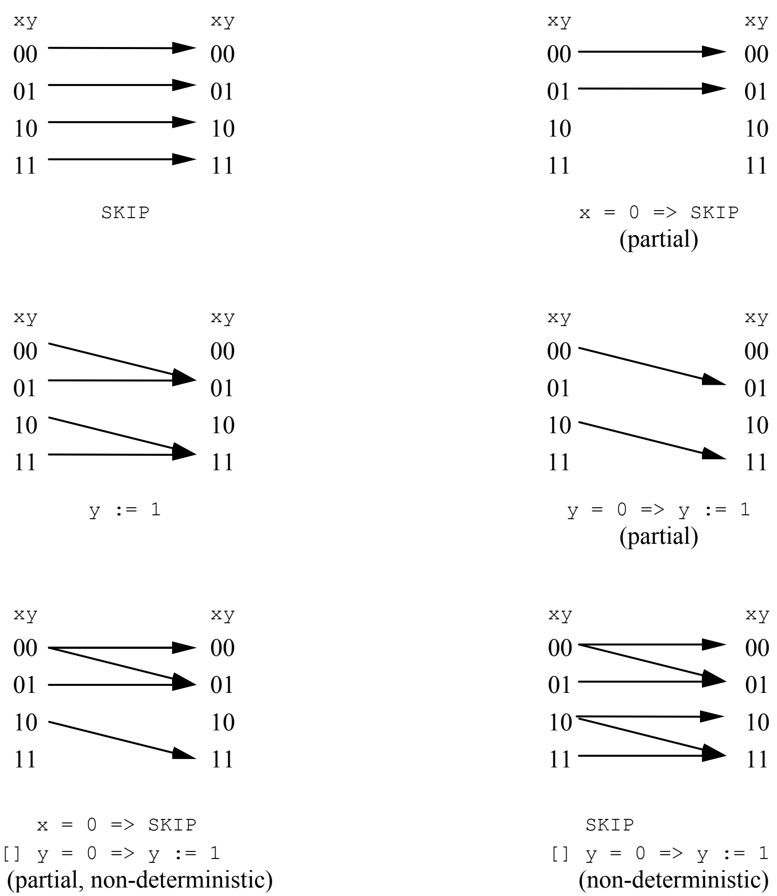
Section 6 of the reference manual describes commands. *Atomic Semantics of Spec* gives a precise account of their semantics. It explains that the meaning of a command is a *relation* between a state and an outcome (a state plus an optional exception), that is, a set of possible state-to-outcome transitions.

Conditionals and choice

Combining commands

The figure below (copied from Nelson's paper) illustrates conditionals and choice with some very simple examples. Here is how they work:

The command
```
p => c
```
means to do `c` if `p` is true. If `p` is false this command fails; in other words, it has no outcome. More precisely, if `s` is a state in which `p` is false or undefined, this command does not relate `s` to any outcome.



What good is such a command? One possibility is that `p` will be true some time in the future, and then the command will have an outcome and allow a transition. Of course this can only happen in a concurrent program, where there is something else going on that can make `p` true. Even if there's no concurrency, there might be an alternative to this command. For instance, it might appear in the larger command


```
     p  => c
[]   p' => c'
```
in which you read `[]` as 'or'. This fails only if each of `p` and `p'` is false or undefined. If both are true (as in the `00` state in the south-west corner of the figure), it means to do either `c` or `c'`; the choice is non-deterministic. If `p'` is `~p` then they are never both false, and if `p` is defined this command is equivalent to
```
     p => c
[*]  c'
```
in which you read `[*]` as 'else'. On the other hand, if `p` is undefined the two commands differ, because the first one fails (since neither guard can be evaluated), while the second does `c'`.

Both `c1 [] c2` and `c1 [*] c2` fail only if *both* `c1` and `c2` fail. If you think of a Spec program operationally (that is, as executing one command after another), this means that if the execution makes some choice that leads to failure later on, it must 'back-track' and try the other alternatives until it finds a set of choices that succeed. For instance, no matter what `x` is, after
```
     y = 0 => x := x - 1; x < y => x := 1
[]   y > 0 => x := 3    ; x < y => x := 2
[*]  SKIP
```
if `y = 0` initially, `x = 1` afterwards, if `y > 3` initially, `x = 2` afterwards, and otherwise `x` is unchanged. If you think of it relationally, `c1 [] c2` has all the transitions of `c1` (there are none if `c1` fails, several if it is non-deterministic) as well as all the transitions of `c2`. Both failure and non-determinism can arise from deep inside a complex command, not just from a top-level `[]` or `VAR`.

This is sometimes called 'angelic' non-determinism, since the code finds all the possible transitions, yielding an outcome if *any* possible non-deterministic choice yield that outcome. This is usually what you want for a spec or high-level code; it is not so good for low-level code, since an operational implementation requires backtracking. The other kind of non-determinism is called 'demonic'; it yields an outcome only if *all* possible non-deterministic choice yield that outcome. To do a command `C` and check that all outcomes satisfy some predicate `p`, write `<< C; ~ p => abort >> [*] C`. The command before the `[*]` does `abort` if some outcome does not satisfy `p`; if every outcome satisfies `p` it fails (doing nothing), and the else clause does `C`.

The precedence rules for commands are

`EXCEPT`	binds tightest
`;`	next
`=> |`	next (for the right operand; the left side is an expression or delimited by `VAR`)
`[] [*]`	bind least tightly.

These rules minimize the need for parentheses, which are written around commands in the ugly form `BEGIN ... END` or the slightly prettier form `IF ... FI`; the two forms have the same meaning, but as a matter of style, the latter should only be used around guarded commands. So, for example,
```
p => c1; c2
```
is the same as
```
p => BEGIN c1; c2 END
```
and means to do `c1` followed by `c2` if `p` is true. To guard only `c1` with `p` you must write
```
IF p => c1 [*] SKIP FI; c2
```
which means to do `c1` if `p` is true, and then to do `c2`. The `[*] SKIP` ensures that the command before the `";"` does not fail, which would prevent `c2` from getting done. Without the `[*] SKIP`, that is in
```
IF p => c1 FI; c2
```

if `p` is false the `IF ... FI` fails, so there is no possible outcome from which `c2` can be done and the whole thing fails. Thus `IF p => c1 FI; c2` has the same meaning as `p => BEGIN c1; c2 END`, which is a bit surprising.

Sequencing

A `c1 ; c2` command means just what you think it does: first `c1`, then `c2`. The command `c1 ; c2` gets you from state `s1` to state `s2` if there is an intermediate state `s` such that `c1` gets you from `s1` to `s` and `c2` gets you from `s` to `s2`. In other words, its relation is the composition of the relations for `c1` and `c2`; sometimes '`;`' is called 'sequential composition'. If `c1` produces an exception, the composite command ignores `c2` and produces that exception.

A `c1 EXCEPT ex => c2` command is just like `c1 ; c2` except that it treats the exception `ex` the other way around: if `c1` produces the exception `ex` then it goes on to `c2`, but if `c1` produces a normal outcome (or any other exception), the composite command ignores `c2` and produces that outcome.

Variable introduction

`VAR` gives you more dramatic non-determinism than `[]`. The most common use is in the idiom

```
VAR x: T | P(x) => c
```

which is read "choose some `x` of type `T` such that `P(x)`, and do `c`". It fails if there is no `x` for which `P(x)` is true and `c` succeeds. If you just write

```
VAR x: T | c
```

then `VAR` acts like ordinary variable declaration, giving an arbitrary initial value to `x`.

Variable introduction is an alternative to existential quantification that lets you get your hands on the bound variable. For instance, you can write

```
IF   VAR n: Nat, x: Nat, y: Nat, z: Nat |
     (n > 2 /\ x**n + y**n = z**n) => out := n
[*]  out := 0
FI
```

which is read: choose integers `n, x, y, z` such that `n > 2` and $x^n + y^n = z^n$, and assign `n` to `out`; if there are no such integers, assign `0` to `out`.[6] The command before the `[*]` succeeds iff

```
(EXISTS n: Int, x: Int, y: Int, z: Int | n > 2 /\ x**n + y**n = z**n),
```

but if we wrote that in a guard there would be no way to set `out` to one of the `n`'s that exist. We could also write

```
VAR s := { n: Int, x: Int, y: Int, z: Int
           | n > 2 /\ x**n + y**n = z**n
          || (n, x, y, z)}
```

to construct the set of all solutions to the equation. Then if `s # {}`, `s.choose` yields a tuple `(n, x, y, z)` with the desired property.

You can use `VAR` to describe all the transitions to a state that has an arbitrary relation `R` to the current state: `VAR s' | R(s, s') => s := s'` if there is only one state variable `s`.

The precedence of `|` is higher than `[]`, which means that you can string together different `VAR` commands with `[]` or `[*]`, but if you want several alternatives within a `VAR` you have to use `BEGIN ... END` or `IF ... FI`. Thus

[6] A correctness proof for an implementation of this spec defied the best efforts of mathematicians between Fermat's time and 1993.


```
      VAR x: T | P(x) => c1
[]    q => c2
```

is parsed the way it is indented and is the same as

```
      BEGIN VAR x: T | P(x) => c1 END
[]    BEGIN q => c2 END
```

but you must write the brackets in

```
VAR x: T |
    IF P(x) => c1
    [] Q(x) => c2
    FI
```

which might be formatted more concisely as

```
VAR x: T |
    IF   P(x) => c1
    []   R(x) => c2 FI
```

or even

```
VAR x: T | IF P(x) => c1 [] R(x) => c2 FI
```

You are supposed to indent your programs to make it clear how they are parsed.

Loops

You can always write a recursive routine, but sometimes a loop is clearer . In Spec you use `DO ... OD` for this. These are brackets, and the command inside is repeated as long as it succeeds. When it fails, the repetition is over and the `DO ... OD` is complete. The most common form is

```
DO P => C OD
```

which is read "while `P` is true do `C`". After this command, `P` must be false. If the command inside the `DO ... OD` succeeds forever, the outcome is a looping exception that cannot be handled. Note that this is not the same as a failure, which means no outcome at all.

For example, you can zero all the elements of a sequence `s` with

```
VAR i := 0 | DO i < s.size => s(i) := 0; i - := 1 OD
```

or the simpler form (which also avoids fixing the order of the assignments)

```
DO VAR i | s(i) # 0 => s(i) := 0 OD
```

This is another common idiom: keep choosing an `i` as long as you can find one that satisfies some predicate. Since `s` is only defined for `i` between `0` and `s.size-1`, the guarded command fails for any other choice of `i`. The loop terminates, since the `s(i) := 0` definitely reduces the number of `i`'s for which the guard is true. But although this is a good example of a loop, it is bad style; you should have used a sequence method or function composition:

```
s := 0.fill(s.size)
```

or

```
s := {x :IN s || 0}
```

(a sequence just like `s` except that every element is mapped to `0`), remembering that Spec makes it easy to throw around big things. Don't write a loop when a constructor will do, because the loop is more complicated to think about. Even if you are writing code, you still shouldn't use a loop here, because it's quite clear how to write C code for the constructor.

To zero all the elements of `s` that satisfy some predicate `P` you can write

```
DO VAR i: Int | (s(i) # 0 /\ P(s(i))) => s(i) := 0 OD
```

Again, you can avoid the loop by using a sequence constructor and a conditional expression

```
s := {x :IN s || (P(x) => 0 [*] x) }
```

Atomicity

Each `<<...>>` command is atomic. It defines a single transition, which includes moving the program counter (which is part of the state) from before to after the command. If a command is not inside `<<...>>`, it is atomic only if there's no reasonable way to split it up: `SKIP`, `HAVOC`, `RET`, `RAISE`. Here are the reasonable ways to split up the other commands:

- An assignment has one internal program counter value, between evaluating the right hand side expression and changing the left hand side variable.
- A guarded command likewise has one, between evaluating the predicate and the rest of the command.
- An invocation has one after evaluating the arguments and before the body of the routine, and another after the body of the routine and before the next transition of the invoking command.

Note that evaluating an expression is always atomic.

Modules and names

Spec's modules are very conventional. Mostly they are for organizing the name space of a large program into a two-level hierarchy: `module.id`. It's good practice to declare everything except a few names of global significance inside a module. You can also declare `CONST`'s, just like `VAR`'s.

```
MODULE foo EXPORT i, j, Fact =

CONST c := 1

VAR i := 0
    j := 1

FUNC Fact(n: Int) -> Int =
     IF   n <= 1 => RET 1
     [*]  RET n * Fact(n - 1)
     FI

END foo
```

You can declare an identifier `id` outside of a module, in which case you can refer to it as `id` everywhere; this is short for `Global.id`, so `Global` behaves much like an extra module. If you declare `id` at the top level in module `m`, `id` is short for `m.id` inside of `m`. If you include it in `m`'s `EXPORT` clause, you can refer to it as `m.id` everywhere. All these names are in the *global* state and are shared among all the atomic actions of the program. By contrast, names introduced by a declaration inside a routine are in the *local* state and are accessible only within their scope.

The purpose of the `EXPORT` clause is to define the external interface of a module. This is important because module `T` implements module `S` iff `T`'s behavior at its external interface is a subset of `S`'s behavior at its external interface.

The other feature of modules is that they can be parameterized by types in the same style as `CLU` clusters. The memory systems modules in handout 5 are examples of this.

You can also declare a *class*, which is a module that can be instantiated many times. The `Obj` class produces a global `Obj` type that has as its methods the exported names of the class plus a `new` procedure that returns a new, initialized instance of the class. It also produces a `ObjMod`

module that contains the declaration of the `Obj` type, the code for the methods, and a state variable indexed by `Obj` that holds the state records of the objects. In a method you can refer to the current object instance by `self`. For example:

```
CLASS Stat EXPORT add, mean, variance, reset =

VAR n          :  Int := 0
    sum        :  Int := 0
    sumsq      :  Int := 0

PROC add(i: Int) = n + := 1; sum + := i; sumsq + := i**2
FUNC mean() -> Int = RET sum/n
FUNC variance() -> Int = RET sumsq/n – self.mean**2
PROC reset() = n := 0; sum := 0; sumsq := 0

END Stat
```

Then you can write

```
    VAR s: Stat | s := s.new(); s.add(x); s.add(y); Print(s.variance)
```

In abstraction functions and invariants we also write `obj.n` for field `n` in `obj`'s state.

Section 7 of the reference manual deals with modules. Section 8 summarizes all the uses of names and the scope rules. Section 9 gives several modules used to define the methods of the built-in data types such as functions, sets, and sequences.

This completes the language summary; for more details and greater precision consult the reference manual. The rest of this handout consists of three extended examples of specs and code written in Spec: topological sort, editor buffers, and a simple window system.

Example: Topological sort

Suppose we have a directed graph whose `n+1` vertexes are labeled by the integers `0 .. n`, represented in the standard way by a relation `g`; `g(v1, v2)` is true if `v2` is a successor of `v1`, that is, if there is an edge from `v1` to `v2`. We want a topological sort of the vertexes, that is, a sequence that is a permutation of `0 .. n` in which `v2` follows `v1` whenever `v2` is a successor of `v1` in the relation `g`. Of course this possible only if the graph is acyclic.

```
MODULE TopologicalSort EXPORT V, G, Q, TopSort =

TYPE V = IN 0 .. n                                    % Vertex
     G = (V, V) -> Bool                               % Graph
     Q = SEQ V

PROC TopSort(g) -> Q RAISES {cyclic} =
     IF   VAR q | q IN (0 .. n).perms /\ IsTSorted(q, g) => RET q
     [*]  RAISE cyclic                                % g must be cyclic
     FI

FUNC IsTSorted(q, g) -> Bool =
% Not tsorted if v2 precedes v1 in q but is also a child
     RET ~ (EXISTS v1 :IN q.dom, v2 :IN q.dom | v2 < v1 /\ g(q(v1), q(v2))

END TopologicalSort
```

Note that this solution checks for a cyclic graph. It allows any topologically sorted result that is a permutation of the vertexes, because the `VAR q` in `TopSort` allows any `q` that satisfies the two conditions. The `perms` method on sets and sequences is defined in section 9 of the reference manual; the `dom` method gives the domain of a function. `TopSort` is a procedure, not a function, because its result is non-deterministic; we discussed this point earlier when studying `SquareRoot`. Like that one, this spec has no internal state, since the module has no `VAR`. It doesn't need one, because it does all its work on the input argument.

The following code is from Cormen, Leiserson, and Rivest. It adds vertexes to the front of the output sequence as depth-first search returns from visiting them. Thus, a child is added before its parents and therefore appears after them in the result. Unvisited vertexes are `white`, nodes being visited are `grey`, and fully visited nodes are `black`. Note that all the descendants of a `black` node must be `black`. The `grey` state is used to detect cycles: visiting a `grey` node means that there is a cycle containing that node.

This module has state, but you can see that it's just for convenience in programming, since it is reset each time `TopSort` is called.

```
MODULE TopSortImpl EXPORT V, G, Q, TopSort =       % implements TopSort

TYPE Color = ENUM[white, grey, black]              % plus the spec's types

VAR  out   : Q
     color: V -> Color                             % every vertex starts white

PROC TopSort(g) -> Q RAISES {cyclic} = VAR i := 0 |
     out := {}; color := {* -> white}
     DO VAR v | color(v) = white => Visit(v, g) OD; % visit every unvisited vertex
     RET out

PROC Visit(v, g) RAISES {cyclic} =
     color(v) := grey;
     DO VAR v' | g(v, v') /\ color(v') # black =>  % pick an successor not done
          IF   color(v') = white => Visit(v', g)
          [*]  RAISE cyclic                        % grey — partly visited
          FI
     OD;
     color(v) := black; out := {v} + out           % add v to front of out
```

The code is as non-deterministic as the spec: depending on the order in which `TopSort` chooses `v` and `Visit` chooses `v'`, any topologically sorted sequence can result. We could get deterministic code in many ways, for example by using `min` to take the smallest node in each case:

```
     VAR v  := {v0 | color(v0) = white}.min              in TopSort
     VAR v' := {v0 | g(v, v0) /\ color(v') # black }.min     in Visit
```

Code in C would do something like this; the details would depend on the representation of `G`.

Example: Editor buffers

A text editor usually has a *buffer* abstraction. A buffer is a mutable sequence of `C`'s. To get started, suppose that `C = Char` and a buffer has two operations,

`Get(i)` to get character `i`

`Replace` to replace a subsequence of the buffer by a subsequence of an argument of type `SEQ C`, where the subsequences are defined by starting position and size.

We can make this spec precise as a Spec class.


```
CLASS Buffer EXPORT B, C, X, Get, Replace =

TYPE X = Nat                                        % indeX in buffer
     C = Char
     B = SEQ C                                      % Buffer contents

VAR  b : B := {}                                    % Note: initially empty

FUNC Get(x) -> C = RET b(x)                         % Note: defined iff 0<=x<b.size

PROC Replace(from: X, size: X, b': B, from': X, size': X) =
% Note: fails if it touches C's that aren't there.
     VAR b1, b2, b3 | b = b1 + b2 + b3 /\ b1.size = from /\ b2.size = size =>
       b := b1 + b'.seg(from', size') + b3

END Buffer
```

We can implement a buffer as a sorted array of *pieces* called a 'piece table'. Each piece contains a `SEQ C`, and the whole buffer is the concatenation of all the pieces. We use binary search to find a piece, so the cost of `Get` is at most logarithmic in the number of pieces. `Replace` may require inserting a piece in the piece table, so its cost is at most linear in the number of pieces.[7] In particular, neither depends on the number of `C`'s. Also, each `Replace` increases the size of the array of pieces by at most two.

A piece is a `B` (in C it would be a pointer to a `B`) together with the sum of the length of all the previous pieces, that is, the index in `Buffer.b` of the first `C` that it represents; the index is there so that the binary search can work. There are internal routines `Locate(x)`, which uses binary search to find the piece containing `x`, and `Split(x)`, which returns the index of a piece that starts at `x`, if necessary creating it by splitting an existing piece. `Replace` calls `Split` twice to isolate the substring being removed, and then replaces it with a single piece. The time for `Replace` is linear in `pt.size` because on the average half of `pt` is moved when `Split` or `Replace` inserts a piece, and in half of `pt`, `p.x` is adjusted if `size' # size`.

```
CLASS BufImpl EXPORT B,C,X, Get, Replace =          % implements Buffer

TYPE                                                % Types as in Buffer, plus
     N   = X                                        % iNdex in piece table
     P   = [b, x]                                   % Piece: x is pos in Buffer.b
     PT  = SEQ P                                    % Piece Table

VAR  pt  := PT{}

ABSTRACTION FUNCTION buffer.b = + : {p :IN pt || p.b}
% buffer.b is the concatenation of the contents of the pieces in pt

INVARIANT (ALL n :IN pt.dom |  pt(n).b # {}
                             /\ pt(n).x = + :{i :IN 0 .. n-1 || pt(i).b.size})
% no pieces are empty, and x is the position of the piece in Buffer.b, as promised.

FUNC Get(x) -> C = VAR p := pt(Locate(x)) | RET p.b(x - p.x)

PROC Replace(from: X, size: X, b': B, from': X, size': X) =
     VAR n1 := Split(from); n2 := Split(from + size),
         new := P{b := b'.seg(from', size'), x := from} |
```

[7] By using a tree of pieces rather than an array, we could make the cost of `Replace` logarithmic as well, but to keep things simple we won't do that. See `FSImpl` in handout 7 for more on this point.

```
        pt :=  pt.sub(0, n1 - 1)
             + NonNull(new)
             + pt.sub(n2, pt.size - 1) * AdjustX(size' - size )

PROC Split(x) -> N =
% Make pt(n)  start at x, so pt(Split(x)).x = x.  Fails if x > b.size.
% If  pt=abcd|efg|hi, then Split(4)  is  RET 1  and  Split(5)  is  pt:=abcd|e|fg|hi; RET 2
    IF   pt = {} /\ x = 0 => RET 0
    [*] VAR n := Locate(x), p := pt(n), b1, b2 |
        p.b = b1 + b2 /\ p.x + b1.size = x =>
             VAR frag1 := p{b := b1}, frag2 := p{b := b2, x := x} |
                  pt :=  pt.sub(0, n - 1)
                       + NonNull(frag1) + NonNull(frag2)
                       + pt.sub(n + 1, pt.size - 1);
                  RET (b1 = {} => n [*] n + 1)
    FI

FUNC Locate(x) -> N = VAR n1 := 0, n2 := pt.size - 1 |
% Use binary search to find the piece containing x. Yields 0 if pt={},
% pt.size-1 if pt#{} /\ x>=b.size; never fails.  The loop invariant is
%   pt={} \/ n2 >= n1 /\ pt(n1).x <= x /\ ( x < pt(n2).x \/ x >= pt.last.x )
% The loop terminates because n2 - n1 > 1 ==> n1 < n < n2, so n2 - n1 decreases.
    DO n2 - n1 > 1 =>
         VAR n := (n1 + n2)/2 | IF pt(n).x <= x => n1 := n [*] n2 := n FI
    OD; RET (x < pt(n2).x => n1 [*] n2)

FUNC NonNull(p) -> PT = RET (p.b # {} => PT{p} [*] {})

FUNC AdjustX(dx: Int) -> (P -> P) = RET (\ p | p{x + := dx})

END BufImpl
```

If subsequences were represented by their starting and ending positions, there would be lots of extreme cases to worry about.

Suppose we now want each `C` in the buffer to have not only a character code but also some additional properties, for instance the font, size, underlining, etc; that is, we are changing the definition of `C` to include the new properties. `Get` and `Replace` remain the same. In addition, we need a third exported method `Apply` that applies to each character in a subsequence of the buffer a `map` function `C -> C`. Such a function might make all the `C`'s italic, for example, or increase the font size by 10%.

```
PROC Apply(map: C->C, from: X, size: X) =
    b :=   b.sub(0, from-1)
         + b.seg(from, size) * map
         + b.sub(from + size, b.size-1)
```

Here is code for `Apply` that takes time linear in the number of pieces. It works by changing the representation to add a `map` function to each piece, and in `Apply` composing the `map` argument with the `map` of each affected piece. We need a new version of `Get` that applies the proper `map` function, to go with the new representation.

```
TYPE P   = [b, x, map: C->C]                          % x is pos in Buffer.b

ABSTRACTION FUNCTION buffer.b = + :{p :IN pt || p.b * p.map}
% buffer.b is the concatenation of the pieces in p with their map's applied.
% This is the same AF we had before, except for the addition of * p.map.

FUNC Get(x) -> C = VAR p := pt(Locate(x)) | RET p.map(p.b(x - p.x))
```



```
PROC Apply(map: C->C, from: X, size: X) =
  VAR n1 := Split(from), n2 := Split(from + size) |
    pt :=  pt.sub(0 , n1 - 1)
         + pt.sub(n1, n2 - 1) * (\ p | p{map := p.map * map})
         + pt.sub(n2, pt.size - 1)
```

Note that we wrote `Split` so that it keeps the same `map` in both parts of a split piece. We also need to add `map := (\ c | c)` to the constructor for `new` in `Replace`.

This code was used in the Bravo editor for the Alto, the first what-you-see-is-what-you-get editor. It is still used in Microsoft Word.

Example: Windows

A window (the kind on your computer screen, not the kind in your house) is a map from points to colors. There can be lots of windows on the screen; they are ordered, and closer ones block the view of more distant ones. Each window has its own coordinate system; when they are arranged on the screen, an offset says where each window's origin falls in screen coordinates.

```
MODULE Window EXPORT Get, Paint =

TYPE I         = Int
     Coord     = Nat
     Intensity = IN (0 .. 255).rng
     P         = [x: Coord, y: Coord] WITH {"-":=PSub}       % Point
     C         = [r: Intensity, g: Intensity, b: Intensity]  % Color
     W         = P -> C                                      % Window

FUNC PSub(p1, p2) -> P = RET P{x := p1.x - p2.x, y := p1.y - p2.y}
```

The shape of the window is determined by the points where it is defined; obviously it need not be rectangular in this very general system. We have given a point a "–" method that computes the vector distance between two points; we somewhat confusingly represent the vector as a point.

A 'window system' consists of a sequence of `[w, offset: P]` pairs; we call a pair a `V`. The sequence defines the ordering of the windows (windows closer to the top come first in the sequence); it is indexed by 'window number' `WN`. The `offset` gives the screen coordinate of the window's `(0, 0)` point, which we think of as its upper left corner. There are two main operations: `Paint(wn, p, c)` to set the value of `P` in window `wn`, and `Get(p)` to read the value of `p` in the topmost window where it is defined (that is, the first one in the sequence). The idea is that what you see (the result of `Get`) is the result of painting the windows from last to first, offsetting each one by its `offset` component and using the color that is painted later to completely overwrite one painted earlier. Of course real window systems have other operations to change the shape of windows, add, delete, and move them, change their order, and so forth, as well as ways for the window system to suggest that newly exposed parts of windows be repainted, but we won't consider any of these complications.

First we give the spec for a window system initialized with `n` empty windows. It is customary to call the coordinate system used by `Get` the *screen* coordinates. The `v.offset` field gives the screen coordinate that corresponds to `{0, 0}` in `v.w`. The `v.c(p)` method below gives the value of `v`'s window at the point corresponding to `p` after adjusting by `v`'s offset. The state `ws` is just the sequence of `V`'s. For simplicity we initialize them all with the same offset `{10, 5}`, which is not too realistic.

`Get` finds the smallest `WN` that is defined at `p` and uses that window's color at `p`. This corresponds to painting the windows from last (biggest `WN`) to first with opaque paint, which is what we wanted. `Paint` uses window rather than screen coordinates.

The state (the `VAR`) is a single sequence of windows on the screen, called `V`'s..

```
TYPE WN          =  IN 0 .. n-1                              % Window Number
     V           =  [w, offset: P]                           % window on the screen
                    WITH {c:=(\ v, p | v.w(p - v.offset))}   % C of a screen point p

VAR  ws: SEQ V   := {i :IN 0..n-1 || V{{}, P{10,5}}}         % the Window System

FUNC Get(p) -> C = VAR wn := {wn' | V.c!(ws(wn'), p)}.min | RET ws(wn).c(p)

PROC Paint(wn, p, c) = ws(wn).w(p) := c

END Window
```

Now we give code that only keeps track of the visible color of each point (that is, it just keeps the pixels on the screen, not all the pixels in windows that are covered up by other windows). We only keep enough state to handle `Get` and `Paint`, so in this code windows can't move or get smaller. In a real window system an "expose" event tells a window to deliver the color of points that become newly visible.

The state is one `W` that represents the screen, plus an `exposed` variable that keeps track of which window is exposed at each point, and the offsets of the windows. This is sufficient to implement `Get` and `Paint`; to deal with erasing points from windows we would need to keep more information about what other windows are defined at each point, so that `exposed` would have a type `P -> SET WN`. Alternatively, we could keep track for each window of where it is defined. Real window systems usually do this, and represent `exposed` as a set of visible regions of the various windows. They also usually have a 'background' window that covers the whole screen, so that every point on the screen has some color defined; we have omitted this detail from the spec and the code.

We need a history variable `wH` that contains the `w` part of all the windows. The abstraction function just combines `wH` and `offset` to make `ws`. Note that the abstract state `ws` is a sequence, that is, a function from window number to `V` for the window. The abstraction function gives the value of the `ws` function in terms of the code variables `wH` and `offset`; that is, it is a function from `wH` and `offset` to `ws`. By convention, we don't write this as a function explicitly.

The important properties of the code are contained in the invariant, from which it's clear that `Get` returns the answer specified by `Window.Get`. Another way to do it is to have a history variable `wsH` that is equal to `ws`. This makes the abstraction function very simple, but then we need an invariant that says `offset(wn) = wsH(n).offset`. This is perfectly correct, but it's usually better to put as little stuff in history variables as possible.

```
MODULE WinImpl EXPORT Get, Paint =

VAR  w        :=  W{}                                   % no points defined
     exposed :    P -> WN := {}                         % which wn shows at p
     offset   :=  {i :IN 0..n-1 || P(5, 10)}            %
     wH       :=  {i :IN 0..n-1 || W{}}                 % history variable

ABSTRACTION FUNCTION ws = (\ wn | V{w := wH(wn), offset := offset(wn)})
```



```
INVARIANT
    (ALL p |    w!p = exposed!p
           /\ (w!p ==>    {wn | V.c!(ws(wn), p)}.min = exposed(p)
                      /\ w(p) = ws(exposed(p)).c(p) ) )
```

The invariant says that each visible point comes from some window, `exposed` tells the topmost window that defines it, and its color is the color of the point in that window. Note that for convenience the invariant uses the abstraction function; of course we could have avoided this by expanding it in line, but there is no reason to do so, since the abstraction function is a perfectly good function.

```
FUNC Get(p) -> C = RET w(p)

PROC Paint(wn, p, c) =
    VAR p0 | p = p0 - offset(wn) =>                          % the screen coordinate
      IF wn <= exposed(p0) => w(p0) := c; exposed(p0) := wn [*] SKIP FI;
      wH(wn)(p) := c                                         % update the history var
END WinImpl
```

4. Spec Reference Manual

Spec is a language for writing specifications and the first few stages of successive refinement towards practical code. As a specification language it includes constructs (quantifiers, backtracking or non-determinism, some uses of atomic brackets) which are impractical in final code; they are there because they make it easier to write clear, unambiguous and suitably general specs. If you want to write a practical program, avoid them.

This document defines the syntax of the language precisely and the semantics informally**. You should read the** *Introduction to Spec* (handout 3) **before trying to read this manual**. In fact, this manual is intended mainly for reference; rather than reading it carefully, skim through it, and then use the index to find what you need. For a precise definition of the atomic semantics read *Atomic Semantics of Spec* (handout 9). Handout 17 on *Formal Concurrency* gives the non-atomic semantics semi-formally.

1. Overview

Spec is a notation for writing specs for a discrete system. What do we mean by a spec? It is the allowed sequences of transitions of a state machine. So Spec is a notation for describing sequences of transitions of a state machine.

Expressions and commands

The Spec language has two essential parts:

An *expression* describes how to compute a value as a function of other values, either constants or the current values of state variables.

A *command* describes possible transitions, or changes in the values of the state variables.

Both are based on the *state*, which in Spec is a mapping from names to values. The names are called state variables or simply variables: in the examples below they are `i` and `j`.

There are two kinds of commands:

An *atomic* command describes a set of possible transitions. For instance, the command `<< i := i + 1 >>` describes the transitions `i=1`→`i=2, i=2`→`i=3`, etc. (Actually, many transitions are summarized by `i=1`→`i=2`, for instance, `(i=1, j=1)`→`(i=2, j=1)` and `(i=1, j=15)`→`(i=2, j=15)`). If a command allows more than one transition from a given state we say it is *non-deterministic*. For instance, the command, `<< i := 1 [] i := i + 1 >>` allows the transitions `i=2`→`i=1` and `i=2`→`i=3`. More on this in *Atomic Semantics of Spec*.

A *non-atomic* command describes a set of sequences of states. More on this in *Formal Concurrency*.

A sequential program, in which we are only interested in the initial and final states, can be described by an atomic command.

Spec's notation for commands, that is, for changing the state, is derived from Edsger Dijkstra's guarded commands (E. Dijkstra, *A Discipline of Programming*, Prentice-Hall, 1976) as extended

by Greg Nelson (G. Nelson, A generalization of Dijkstra's calculus, *ACM TOPLAS* **11**, 4, Oct. 1989, pp 517-561). The notation for expressions is derived from mathematics.

Organizing a program

In addition to the expressions and commands that are the core of the language, Spec has four other mechanisms that are useful for organizing your program and making it easier to understand.

A *routine* is a named computation with parameters (passed by value). There are four kinds:

A *function* is an abstraction of an expression.

An *atomic procedure* is an abstraction of an atomic command.

A general procedure is an abstraction of a non-atomic command.

A *thread* is the way to introduce concurrency.

A *type* is a stylized assertion about the set of values that a name can assume. A type is also an easy way to group and name a collection of routines, called its *methods*, that operate on values in that set.

An *exception* is a way to report an unusual outcome.

A *module* is a way to structure the name space into a two-level hierarchy. An identifier `i` declared in a module `m` is known as `i` in `m` and as `m.i` throughout the program. A *class* is a module that can be instantiated many times to create many objects.

A Spec program is some global declarations of variables, routines, types, and exceptions, plus a set of modules each of which declares some variables, routines, types, and exceptions.

Outline

This manual describes the language bottom-up:
Lexical rules
Types
Expressions
Commands
Modules

At the end there are two sections with additional information:
Scope rules
Built-in methods for set, sequence, and routine types.

There is also an index. The *Introduction to Spec* has a one-page language summary.

2. Grammar rules

Nonterminal symbols are in lower case; terminal symbols are punctuation other than `::=`, or are quoted, or are in upper case.

Alternative choices for a nonterminal are on separate lines.

`symbol*` denotes zero of more occurrences of `symbol`.

The symbol `empty` denotes the empty string.

If `x` is a nonterminal, the nonterminal `xList` is defined by

```
xList        ::= x
                 x , xList
```

A comment in the grammar runs from `%` to the end of the line; this is just like Spec itself.

A `[n]` in a comment means that there is an explanation in a note labeled `[n]` that follows this chunk of grammar.

3. Lexical rules

The symbols of the language are literals, identifiers, keywords, operators, and the punctuation `( ) [ ] { } , ; : . | << >> := => -> [] [*]`. Symbols must not have embedded white space. They are always taken to be as long as possible.

A *literal* is a decimal number such as `3765`, a quoted character such as `'x'`, or a double-quoted string such as `"Hello\n"`.

An *identifier* (`id`) is a letter followed by any number of letters, underscores, and digits followed by any number of `'` characters. Case is significant in identifiers. By convention type and procedure identifiers begin with a capital letter. An identifier may not be the same as a keyword. The *predefined* identifiers `Any, Bool, Char, Int, Nat, Null, String, true, false`, and `nil` are declared in every program. The meaning of an identifier is established by a declaration; see section 8 on scope for details. Identifiers cannot be redeclared.

By convention *keywords* are written in upper case, but you can write them in lower case if you like; the same strings with mixed case are not keywords, however. The keywords are

ALL	APROC	AS	BEGIN	BY	CLASS
CONST	DO	END	ENUM	EXCEPT	EXCEPTION
EXISTS	EXPORT	FI	FUNC	HAVOC	IF
IN	IS	LAMBDA	MODULE	OD	PROC
RAISE	RAISES	RET	SEQ	SET	SKIP
SUCHTHAT	THREAD	TYPE	VAR	WHILE	WITH

An *operator* is any sequence of the characters `!@#$^&*-+=:.<>?/\|~` except the sequences `: . | << >> := => ->` (these are punctuation), or one of the keyword operators `AS`, `IN`, and `IS`.

A comment in a Spec program runs from a `%` outside of quotes to the end of the line. It does not change the meaning of the program.

4. Types

A type defines a set of values; we say that a value `v` has type `T` if `v` is in `T`'s set. The sets are not disjoint, so a value can belong to more than one set and therefore can have more than one type. In addition to its value set, a type also defines a set of routines (functions or procedures) called its *methods*; a method normally takes a value of the type as its first argument.

An expression has exactly one type, determined by the rules in section 5; the result of the expression has this type unless it is an exception.

The picky definitions given on the rest of this page are the basis for Spec's type-checking. You can skip them on first reading, or if you don't care about type-checking.

About unions: If the expression `e` has type `T` we say that `e` has a routine type `W` if `T` is a routine type `W` or if `T` is a union type and exactly one type `W` in the union is a routine type. Note that this covers sequence, tuple, and record types. Under corresponding conditions we say that `e` has a set type.

Two types are *equal* if their definitions are the same (that is, have the same parse trees) after all type names have been replaced by their definitions and all `WITH` clauses have been discarded. Recursion is allowed; thus the expanded definitions might be infinite. Equal types define the same value set. Ideally the reverse would also be true, but type equality is meant to be decided by a type checker, whereas the set equality is intractable.

A type `T` *fits* a type `U` if the type-checker thinks it's OK to use a `T` where a `U` is required. This is true if the type-checker thinks they may have some non-trivial values in common. This can only happen if they have the same structure, and each part of `T` fits the corresponding part of `U`. 'Fits' is an equivalence relation. Precisely, `T` fits `U` if:

`T = U`.

`T` is `T' SUCHTHAT F` or `(... + T' + ...)` and `T'` fits `U`, or vice versa. There may be no values in common, but the type-checker can't analyze the `SUCHTHAT` clauses to find out. There's a special case for the `SUCHTHAT` clauses of record and tuple types, which the type-checker *can* analyze: `T`'s `SUCHTHAT` must imply `U`'s.

`T=T1->T2 RAISES EXt` and `U=U1->U2 RAISES EXu`, or one or both `RAISES` are missing, and `U1` fits `T1` and `T2` fits `U2`. Similar rules apply for `PROC` and `APROC` types. This also covers sequences. Note that the test is reversed for the argument types.

`T=SET T'` and `U=SET U'` and `T'` fits `U'`.

`T` *includes* `U` if the same conditions apply with "fits" replaced by "includes", all the "vice versa" clauses dropped, and in the `->` rule "`U1` fits `T1`" replaced by "`U1` includes `T1` and `EXt` is a superset of `EXu`". If `T` includes `U` then `T`'s value set includes `U`'s value set; again, the reverse is intractable.

An expression `e` fits a type `U` in state `s` if `e`'s type fits `U` and the result of `e` in state `s` has type `U` or is an exception; in general this can only be checked at runtime unless `U` includes `e`'s type. The check that `e` fits `T` is required for assignment and routine invocation; together with a few other checks it is called *type-checking*. The rules for type-checking are given in sections 5 and 6.

```
type          ::= name                            % name of a type
                  "Any"                           % every value has this type
                  "Null"                          % with value set {nil}
                  "Bool"                          % with value set {true, false}
                  "Char"                          % like an enumeration
                  "String"                        % = SEQ Char
                  "Int"                           % integers
                  "Nat"                           % naturals: non-negative integers
                  SEQ type                        % sequence [1]
                  SET type                        % set
                  [ declList ]                    % record with declared fields [7]
                  ( typeList )                    % tuple; (T) is the same as T [8]
                   ( union )                      % union of the types
                  aType ->  type raises           % function [2]
                  aType ->> type raises           % relation [2]
                  APROC aType returns raises      % atomic procedure [2]
                  PROC  aType returns raises      % non-atomic procedure [2]
                  type WITH { methodDefList }     % attach methods to a type [3]
                  type SUCHTHAT primary           % restrict the value set [4]
                  IN exp                          % = T SUCHTHAT (\ t: T | t IN exp)
                                                  % where exp's type has an IN method
                  id [ typeList ] . id            % type from a module [5]

name          ::= id . id                         % the first id denotes a module
                  id                              % short for m.id if id is declared
                                                  % in the current module m, and for
                                                  % Global.id if id is declared globally
                  type . id                       % the id method of type
decl          ::= id : type                       % id has this type
                  id                              % short for id: Id [6]

union         ::= type  + type
                  union + type

aType         ::= ()
                  type
returns       ::= empty                           % only for procedures
                  -> type
raises        ::= empty
                  RAISES exceptionSet             % the exceptions it can return

exceptionSet  ::= { exceptionList }               % a set of exceptions
                  name                            % declared as an exception set
                  exceptionSet \/ exceptionSet    % set union
                  exceptionSet - exceptionSet     % set difference
exception     ::= id                              % means "id"

method        ::= id
                  stringLiteral                   % the string must be an operator
                                                  % other than "=" or "#" (see section 3)
methodDef     ::= method := name                  % name is a routine
```


The ambiguity of the type grammar is resolved by taking `->` to be right associative and giving `WITH` and `RAISES` higher precedence than `->`.

[1] A `SEQ T` is just a function from `0..size-1` to `T`. That is, it is short for

```
(Int->T) SUCHTHAT (\ f: Int->T | (EXISTS size: Int | f.dom = 0..size-1))
         WITH { see section 9 }.
```

This means that invocation, `!`, and `*` work for a sequence just as they do for any function. In addition, there are many other useful operators on sequences; see section 9. The `String` type is just `SEQ Char`; there are `String` literals, defined in section 5.

[2] A `T->U` value is a partial function from a state and a value of type `T` to a value of type `U`. A `T->U RAISES xs` value is the same except that the function may raise the exceptions in `xs`.

A function or procedure declared with names for the arguments, such as

```
(\ i: Int, s: String | i + StringToInt(x))
```

has a type that ignores the names, `(Int, String)->Int`. However, it also has a method `argNames` that returns the sequence of argument names, `{"i", "s"}` in the example, just like a record. This makes it possible to match up arguments by name, as in the following example.

A database is a set `s` of records. A selection query `q` is a predicate that we want to apply to the records. How do we get from the field names, which are strings, to the argument for `q`? Assume that `q` has an `argNames` method. So if `r IN s`, `q.argNames * r` is the tuple that we want to feed to `q`; `q$(q.argNames * r)` is the query, where `$` is the operator that applies a function to a tuple of its arguments.

[3] We say `m` is a *method* of `T` defined by `f`, and denote `f` by `T.m`, if

`T = T' WITH {..., m := f, ...}` and `m` is an identifier or is `"op"` where `op` is an operator (the construct in braces is a `methodDefList`), or

`T = T' WITH { methodDefList }`, `m` is not defined in `methodDefList`, and `m` is a method of `T'` defined by `f`, or

`T = (... + T' + ...)`, `m` is a method of `T'` defined by `f`, and there is no other type in the union with a method `m`.

There are two special forms for invoking methods: `e1 infixOp e2` or `prefixOp e`, and `e1.id(e2)` or `e.id` or `e.id()`. They are explained in notes [1] and [3] to the expression grammar in the next section. This notation may be familiar from object-oriented languages. Unlike many such languages, Spec makes no provision for varying the method in each object, though it does allow inheritance and overriding.

A method doesn't have to be a routine, though the special forms won't type-check unless the method is a routine. Any method `m` of `T` can be referred to by `T.m`.

If type `U` has method `m`, then the function type `V = T->U` has a *lifted* method `m` that composes `U.m` with `v`, unless `V` already has a `m` method. `V. m` is defined by

```
(\ v | (\ t | v(t).m))
```

so that `v.m = v * U.m`. For example, `{"a", "ab", "b"}.size = {1, 2, 1}`. If `m` takes a second argument of type `W`, then `V.m` takes a second argument of type `VV = T->W` and is defined on the intersection of the domains by applying `m` to the two results. Thus in this case `V.m` is

```
(\ v, vv | (\ t :IN v.dom /\ vv.dom | v(t).m(vv(t))))
```

Lifting also works for relations to `U`, and therefore also for `SET U`. Thus if `R = (T,U)->Bool` and `m` returns type `X`, `R.m` is defined by

```
(\ r | (\ t, x | x IN {u | r(t, u) || u.m}))
```

so that `r.m = r * U.m.rel`. If `m` takes a second argument, then `R.m` takes a second argument of type `RR = T->W`, and `r.m(rr)` relates `t` to `u.m(w)` whenever `r` relates `t` to `u` and `rr` relates `t` to `w`. In other words, `R.m` is defined by

```
(\ r, rr | (\ t, x | x IN {u, w | r(t, u) /\ rr(t, w) || u.m(w)}))
```

If `U` doesn't have a method `m` but `Bool` does, then the lifting is done on the function that defines the relation, so that `r1 \/ r2` is the union of the relations, `r1 /\ r2` the intersection, `r1 - r2` the difference, and `~r` the complement.

[4] In `T SUCHTHAT E`, `E` is short for a predicate on `T`'s, that is, a function `(T -> Bool)`. If the context is `TYPE U = T SUCHTHAT E` and `this` doesn't occur free in `E`, the predicate is `(\ u: T | E)`, where `u` is `U` with the first letter changed to lower-case; otherwise the predicate is `(\ this: T | E)`. The type `T SUCHTHAT E` has the same methods as `T`, and its value set is the values of `T` for which the predicate is true. See section 5 for `primary`.

[5] If a type is defined by `m[typeList].id` and `m` is a parameterized module, the meaning is `m'.id` where `m'` is defined by `MODULE m' = m[typeList] END m'`. See section 7 for a full discussion of this kind of type.

[6] `Id` is the `id` of a type, obtained from `id` by dropping trailing ' characters and digits, and capitalizing the first letter or all the letters (it's an error if these capitalizations yield different identifiers that are both known at this point).

[7] The type of a record is `String->Any SUCHTHAT ...`. The `SUCHTHAT` clauses are of the form `this("f") IS T`; they specify the types of the fields. In addition, a record type has a method called `fields` whose value is the sequence of field names (it's the same for every record). Thus `[f: T, g: U]` is short for

```
String->Any WITH { fields:=(\r: String->Any | (SEQ String){"f", "g"}) }
            SUCHTHAT   this.dom >= {"f", "g"}
                       /\ this("f") IS T /\ this("g") IS U
```

[8] The type of a tuple is `Nat->Any SUCHTHAT ...`. As with records, the `SUCHTHAT` clauses are of the form `this("f") IS T`; they specify the types of the fields. In addition, a tuple type has a method called `fields` whose value is `0..n-1` if the tuple has `n` fields. Thus `(T, U)` is short for

```
Int->Any WITH { fields:=(\r: Int->Any | 0..1) }
            SUCHTHAT   this.dom = 0..1
                       /\ this(0) IS T /\ this(1) IS U
```

Thus to convert a record `r` into a tuple, write `r.fields * r`, and to convert a tuple `t` into a record, write `r.fields.inv * t`.

There is no special syntax for tuple fields, since you can just write `t(2)` and `t(2) := e` to read and write the third field, for example (remember that fields are numbered from 0).

5. Expressions

An expression is a partial function from states to results; results are values or exceptions. That is, an expression computes a result for a given state. The state is a function from names to values. This state is supplied by the command containing the expression in a way explained later. The meaning of an expression (that is, the function it denotes) is defined informally in this section. The meanings of invocations and lambda function constructors are somewhat tricky, and the informal explanation here is supplemented by a formal account in *Atomic Semantics of Spec*. Because expressions don't have side effects, the order of evaluation of operands is irrelevant (but see [5] and [13]).

Every expression has a type. The result of the expression is a member of this type if it is not an exception. This property is guaranteed by the *type-checking* rules, which require an expression used as an argument, the right hand side of an assignment, or a routine result to fit the type of the formal, left hand side, or routine range (see section 4 for the definition of 'fit'). In addition, expressions appearing in certain contexts must have *suitable* types: in `e1(e2)`, `e1` must have a routine type; in `e1+e2`, `e1` must have a type with a `"+"` method, etc. These rules are given in detail in the rest of this section. A union type is suitable if exactly one of the members is suitable. Also, if `T` is suitable in some context, so are `T WITH {... }` and `T SUCHTHAT f`.

An expression can be a literal, a variable known in the scope that contains the expression, or a function invocation. The form of an expression determines both its type and its result in a state:

`literal` has the type and value of the literal.

`name` has the declared type of `name` and its value in the current state, `state("name")`. The form `T.m` (where `T` denotes a type) is also a name; it denotes the `m` method of `T`. Note that if `name` is `id` and `id` is declared in the current module `m`, then it is short for `m.id`.

invocation `f(e)`: `f` must have a function (not procedure) type `U->T RAISES EX` or `U->T` (note that a sequence is a function), and `e` must fit `U`; then `f(e)` has type `T`. In more detail, if `f` has result `rf` and `e` has type `U'` and result `re`, then `U'` must fit `U` (checked statically) and `re` must have type `U` (checked dynamically if `U'` involves a union or `SUCHTHAT`; if the dynamic check fails the result is a fatal error). Then `f(e)` has type `T`.

If either `rf` or `re` is undefined, so is `f(e)`. Otherwise, if either is an exception, that exception is the result of `f(e)`; if both are, `rf` is the result.

If both `rf` and `re` are normal, the result of `rf` at `re` can be:

A normal value, which becomes the result of `f(e)`.

An exception, which becomes the result of `f(e)`. If `rf` is defined by a function body that loops, the result is a special looping exception that you cannot handle.

Undefined, in which case `f(e)` is undefined and the command containing it fails (has no outcome) — failure is explained in section 6.

A function invocation in an expression never affects the state. If the result is an exception, the containing command has an exceptional outcome; for details see section 6.

The other forms of expressions (`e.id`, `constructor`s, prefix and infix operators, combinations, and quantifications) are all syntactic sugar for function invocations, and their results are obtained by the rule used for invocations. There is a small exception for conditionals [5] and for the conditional logical operators `/\`,`\/`, and `==>` that are defined in terms of conditionals [13].

```
exp          ::= primary
                 prefixOp exp                   % [1]
                 exp infixOp exp                % [1]
                 infixOp : exp                  % exp's elements combined by op [2]
                 exp IS type                    % (EXISTS x: type | exp = x)
                 exp AS type                    % error unless (exp IS type) [14]

primary      ::= literal
                 name
                 primary . id                   % method invocation [3] or record field
                 primary arguments              % function invocation
                 constructor
                 ( exp )
                 ( quantif declList | pred )    % /\:{d | p} for ALL, \/ for EXISTS [4]
                 ( pred => exp1 [*] exp2 )      % if pred then exp1 else exp2 [5]
                 ( pred => exp1 )               % undefined if pred is false

literal      ::= intLiteral                     % sequence of decimal digits
                 charLiteral                    % 'x', x a printing character
                 stringLiteral                  % "xxx", with \ escapes as in C

arguments    ::= ( expList )                    % the arg is the tuple (expList)
                 ( )

constructor  ::= { }                            % empty function/sequence/set [6]
                 { expList }                    % sequence/set constructor [6]
                 ( expList )                    % tuple constructor
                 name    { }                    % name denotes a func/seq/set type [6]
                 name    { expList }            % name denotes a seq/set/record type [6]
                 primary { fieldDefList  }      % record constructor [7]
                 primary { exp -> result }      % function or sequence constructor [8]
                 primary { *   -> result }      % function constructor [8]
                 ( LAMBDA signature = cmd )     % function with the local state [9]
                 ( \ declList | exp )           % short for (LAMBDA(d)->T=RET exp) [9]
                 { declList   | pred || exp }   % set constructor [10]
                 { seqGenList | pred || exp }   % sequence constructor [11]

fieldDef     ::= id := exp

result       ::= empty                          % the function is undefined
                 exp                            % the function yields exp
                 RAISE exception                % the function yields exception

seqGen       ::= id := exp BY exp WHILE exp     % sequence generator [11]
                 id :IN exp

pred         ::= exp                            % predicate, of type Bool
quantif      ::= ALL
                 EXISTS
```


		(precedence)	***argument/result types***		***operation***
`infixOp ::=`	`**`	% (8)	`(Int, Int)->Int`		exponentiate
	`*`	% (7)	`(Int, Int)->Int`		multiply
		%	`(T->U, U->V)->(T->V)`	[12]	function composition
	`/`	% (7)	`(Int, Int)->Int`		divide
	`//`	% (7)	`(Int, Int)->Int`		remainder
	`+`	% (6)	`(Int, Int)->Int`		add
		%	`(SEQ T, SEQ T)->SEQ T`	[12]	concatenation
		%	`(T->U, T->U)->(T->U)`	[12]	function overlay
	`-`	% (6)	`(Int, Int)->Int`		subtract
		%	`(SET T, SET T)->SET T`	[12]	set difference;
		%	`(SEQ T, SEQ T)->SEQ T`	[12]	multiset difference
	`!`	% (6)	`(T->U, T)->Bool`	[12]	function is defined
	`!!`	% (6)	`(T->U, T)->Bool`	[12]	func has normal value
	`$`	% (6)	`(T->U, T)->U`	[15]	apply func to tuple
	`..`	% (5)	`(Int, Int)->SEQ Int`	[12]	subrange
	`<=`	% (4)	`(Int, Int)->Bool`		less than or equal
		%	`(SET T, SET T)->Bool`	[12]	subset
		%	`(SEQ T, SEQ T)->Bool`	[12]	prefix
	`<`	% (4)	`(T, T)->Bool, T with <=`		less than
		%	`e1<e2 = (e1<=e2 /\ e1#e2)`		
	`>`	% (4)	`(T, T)->Bool, T with <=`		greater than
		%	`e1>e2 = e2<e1`		
	`>=`	% (4)	`(T, T)->Bool, T with <=`		greater or equal
		%	`e1>=e2 = e2<=e1`		
	`=`	% (4)	`(Any, Any)->Bool`	[1]	equal
	`#`	% (4)	`(Any, Any)->Bool`		not equal
		%	`e1#e2 = ~ (e1=e2)`		
	`<<=`	% (4)	`(SEQ T, SEQ T)->Bool`	[12]	non-contiguous sub-seq
	`IN`	% (4)	`(T, SET T)->Bool`	[12]	membership
	`/\`	% (2)	`(Bool, Bool)->Bool`	[13]	conditional and
		%	`(SET T, SET T)->SET T`	[12]	intersection
	`\/`	% (1)	`(Bool, Bool)->Bool`	[13]	conditional or
		%	`(SET T, SET T)->SET T`	[12]	union
	`==>`	% (0)	`(Bool, Bool)->Bool`	[13]	conditional implies
	`op`	% (5)	not one of the above	[1]	
`prefixOp ::=`	`-`	% (6)	`Int->Int`		negation
	`~`	% (3)	`Bool->Bool`		complement
	`op`	% (5)	not one of the above	[1]	

The ambiguity of the expression grammar is resolved by taking the infixOps to be left associative and using the indicated precedences for the prefixOps and infixOps (with 8 for IS and AS and 5 for : or any operator not listed); higher numbers correspond to tighter binding. The precedence is determined by the operator symbol and doesn't depend on the operand types.

[1] The meaning of prefixOp e is T."prefixOp"(e), where T is e's type, and of e1 infixOp e2 is T1."infixOp"(e1, e2), where T1 is e1's type. The built-in types Int (and Nat with the same operations), Bool, sequences, sets, and functions have the operations given in the grammar. Section 9 on built-in methods specifies the operators for built-in types other than Int and Bool. Special case: e1 IN e2 means T2."IN"(e1, e2), where T2 is e2's type.

Note that the = operator does not require that the types of its arguments agree, since both are Any. Also, = and # cannot be overridden by WITH. To define your own abstract equality, use a different operator such as "==".

[2] The exp must have type SEQ T or SET T. The value is the elements of exp combined into a single value by infixOp, which must be associative and have an identity, and must also be commutative if exp is a set. Thus

```
+ : {i: Int | 0<i /\ i<5 | i**2} = 1 + 4 + 9 + 16 = 30,
```

and if s is a sequence of strings, + : s is the concatenation of the strings. For another example, see the definition of quantifications in [4]. Note that the entire set is evaluated; see [10].

[3] Methods can be invoked by dot notation.

The meaning of e.id or e.id() is T.id(e), where T is e's type.

The meaning of e1.id(e2) is T.id(e1, e2), where T is e1's type.

Section 9 on built-in methods gives the methods for built-in types other than Int and Bool.

[4] A quantification is a conjunction (if the quantifier is ALL) or disjunction (if it is EXISTS) of the pred with the id's in the declList bound to every possible value (that is, every value in their types); see section 4 for decl. Precisely, (ALL d | p) = /\ : {d | p} and (EXISTS d | p) = \/ : {d | p}. All the expressions in these expansions are evaluated, unlike e2 in the expressions e1 /\ e2 and e1 \/ e2 (see [10] and [13]).

[5] A conditional (pred => e1 [*] e2) is not exactly an invocation. If pred is true, the result is the result of e1 even if e2 is undefined or exceptional; if pred is false, the result is the result of e2 even if e1 is undefined or exceptional. If pred is undefined, so is the result; if pred raises an exception, that is the result. If [*] e2 is omitted and pred is false, the result is undefined.

[6] In a constructor {expList} each exp must have the same type T, the type of the constructor is (SEQ T + SET T), and its value is the sequence containing the values of the exps in the given order, which can also be viewed as the set containing these values.

If expList is empty the type is the union of all function, sequence and set types, and the value is the empty sequence or set, or a function undefined everywhere. If desired, these constructors can be prefixed by a name denoting a suitable set or sequence type.

A constructor T{e1, ..., en}, where T is a record type [f1: T1, ..., fn: Tn], is short for a record constructor (see [7]) T{f1:=e1, ..., fn:=en}.

[7] The primary must have a record type, and the constructor has the same type as its primary and denotes the same value except that the fields named in the fieldDefList have the given values. Each value must fit the type declared for its id in the record type. The primary may also denote a record type, in which case any fields missing from the fieldDefList are given arbi-

trary (but deterministic) values. Thus if R=[a: Int, b: Int], R{a := 3, b := 4} is a record of type R with a=3 and b=4, and R{a := 3, b := 4}{a := 5} is a record of type R with a=5 and b=4. If the record type is qualified by a SUCHTHAT, the fields get values that satisfy it, and the constructor is undefined if that's not possible.

[8] The primary must have a function or sequence type, and the constructor has the same type as its primary and denotes a value equal to the value denoted by the primary except that it maps the argument value given by exp (which must fit the domain type of the function or sequence) to result (which must fit the range type if it is an exp). For a function, if result is empty the constructed function is undefined at exp, and if result is RAISE exception, then exception must be in the RAISES set of primary's type. For a sequence result must not be empty or RAISE, and exp must be in primary.dom or the constructor expression is undefined.

In the * form the primary must be a function type or a function, and the value of the constructor is a function whose result is result at every value of the function's domain type (the type on the left of the ->). Thus if F=(Int->Int) and f=F{*->0}, then f is zero everywhere and f{4->1} is zero except at 4, where it is 1. If this value doesn't have the function type, the constructor is undefined; this can happen if the type has a SUCHTHAT clause. For example, the type can't be a sequence.

[9] A LAMBDA constructor is a statically scoped function definition. When it is invoked, the meaning of the body is determined by the local state when the LAMBDA was evaluated and the global state when it is invoked; this is ad-hoc but convenient. See section 7 for signature and section 6 for cmd. The returns in the signature may not be empty. Note that a function can't have side effects.

The form (\ declList | exp) is short for (LAMBDA (declList) -> T = RET exp), where T is the type of exp. See section 4 for decl.

[10] A set constructor { declList | pred || exp } has type SET T, where exp has type T in the current state augmented by declList; see section 4 for decl. Its value is a set that contains x iff (EXISTS declList | pred /\ x = exp). Thus

```
{i: Int | 0<i /\ i<5 || i**2} = {1, 4, 9, 16}
```

and both have type SET Int. If pred is omitted it defaults to true. If | exp is omitted it defaults to the last id declared:

```
{i: Int | 0<i /\ i<5} = {1, 2, 3, 4 }
```

Note that if s is a set or sequence, IN s is a type (see section 4), so you can write a constructor like {i :IN s | i > 4} for the elements of s greater than 4. This is shorter and clearer than

```
{i | i IN s /\ i > 4}
```

If there are any values of the declared id's for which pred is undefined, or pred is true and exp is undefined, then the result is undefined. If nothing is undefined, the same holds for exceptions; if more than one exception is raised, the result exception is an arbitrary choice among them.

[11] A sequence constructor { seqGenList | pred || exp } has type SEQ T, where exp has type T in the current state augmented by seqGenList, as follows. The value of

```
{x1 := e01 BY e1 WHILE p1, ... , xn := e0n BY en WHILE pn | pred || exp}
```

is the sequence which is the value of result produced by the following program. Here exp has type T and result is a fresh identifier (that is, one that doesn't appear elsewhere in the program). There's an informal explanation after the program.

```
VAR x2 := e02, ..., xn := e0n, result := T{}, x1 := e01 |
    DO p1 => x2 := e2; p2 => ... => xn := en; pn =>
```

```
            IF pred => result := result + {exp} [*] SKIP FI;
            x1 := e1
        OD
```

However, `e0i` and `ei` are not allowed to refer to `xj` if `j > i`. Thus the `n` sequences are unrolled in parallel until one of them ends, as follows. All but the first are initialized; then the first is initialized and all the others computed, then all are computed repeatedly. In each iteration, once all the `xi` have been set, if `pred` is true the value of `exp` is appended to the result sequence; thus `pred` serves to filter the result. As with set constructors, an omitted `pred` defaults to `true`, and an omitted `|| exp` defaults to `|| xn`. An omitted `WHILE pi` defaults to `WHILE true`. An omitted `:= e0i` defaults to

```
    := {x: Ti | true}.choose
```

where `Ti` is the type of `ei`; that is, it defaults to an arbitrary value of the right type.

The generator `xi :IN ei` generates the elements of the sequence `ei` in order. It is short for

```
    j := 0 BY j + 1 WHILE j < ei.size, xi BY ei(j)
```

where `j` is a fresh identifier. Note that if the `:IN` isn't the first generator then the first element of `ei` is skipped, which is probably not what you want. Note that `:IN` in a sequence constructor overrides the normal use of `IN s` as a type (see [10]).

Undefined and exceptional results are handled the same way as in set constructors.

Examples

`{i := 0 BY i+1 WHILE i <= n}`	`= 0..n = {0, 1, ..., n}`
`(r := head BY r.next WHILE r # nil || r.val}`	the `val` fields of a list starting at `head`
`{x :IN s, sum := 0 BY sum + x}`	partial sums of `s`
`{x :IN s, sum := 0 BY sum + x}.last`	`+ : s`, the last partial sum
`{x :IN s, rev := {} BY {x} + rev}.last`	reverse of `s`
`{x :IN s || f(x)}`	`s * f`
`{i :IN 1..n | i // 2 # 0 || i * i}`	squares of odd numbers `<= n`
`{i :IN 1..n, iter := e BY f(iter)}`	{f(e), f^2(e), ..., f^n(e)}

[12] These operations are defined in section 9.

[13] The conditional logical operators are defined in terms of conditionals:

```
    e1 \/  e2 = (  e1 => true  [*] e2 )
    e1 /\  e2 = ( ~e1 => false [*] e2 )
    e1 ==> e2 = ( ~e1 => true  [*] e2 )
```

Thus the second operand is not evaluated if the value of the first one determines the result.

[14] `AS` changes only the type of the expression, not its value. Thus if `(exp IS type)` the value of `(exp AS type)` is the value of `exp`, but its type is `type` rather than the type of `exp`.

[15] `f$t` applies the function `f` to the tuple `t`. It differs from `f(t)`, which makes a tuple out of the list of expressions in `t` and applies `f` to that tuple.

6. Commands

A command changes the state (or does nothing). Recall that the state is a mapping from names to values; we denote it by `state`. Commands are non-deterministic. An atomic command is one that is inside `<<...>>` brackets.

The meaning of an atomic command is a set of possible transitions (that is, a relation) between a state and an outcome (a state plus an optional exception); there can be any number of outcomes from a given state. One possibility is a looping exceptional outcome. Another is no outcomes. In this case we say that the atomic command *fails*; this happens because all possible choices within it encounter a false guard or an undefined invocation.

If a subcommand fails, an atomic command containing it may still succeed. This can happen because it's one operand of `[]` or `[*]` and the other operand succeeds. If can also happen because a non-deterministic construct in the language that might make a different choice. Leaving exceptions aside, the commands with this property are `[]`and `VAR` (because it chooses arbitrary values for the new variables). If we gave an operational semantics for atomic commands, this situation would correspond to backtracking. In the relational semantics that we actually give (in *Atomic Semantics of Spec*), it corresponds to the fact that the predicate defining the relation is the "or" of predicates for the subcommands. Look there for more discussion of this point.

A non-atomic command defines a collection of possible transitions, roughly one for each `<<...>>` command that is part of it. If it has simple commands not in atomic brackets, each one also defines a possible transition, except for `assignment`s and `invocation`s. An `assignment` defines two transitions, one to evaluate the right hand side, and the other to change the value of the left hand side. An `invocation` defines a transition for evaluating the arguments and doing the call and one for evaluating the result and doing the return, plus all the transitions of the body. These rules are somewhat arbitrary and their details are not very important, since you can always write separate commands to express more transitions, or atomic brackets to express fewer transitions. The motivation for the rules is to have as many transitions as possible, consistent with the idea that an expression is evaluated atomically.

A complete collection of possible transitions defines the possible sequences of states or histories; there can be any number of histories from a given state. A non-atomic command still makes choices, but it does not backtrack and therefore can have histories in which it gets stuck, even though in other histories a different choice allows it to run to completion. For the details, see handout 17 on formal concurrency.

```
cmd          ::= SKIP                                % [1]
                 HAVOC                               % [1]
                 RET                                 % [2]
                 RET exp                             % [2]
                 RAISE exception                     % [9]
                 CRASH                               % [10]
                 invocation                          % [3]
                 assignment                          % [4]

                 cmd                 []   cmd        % or [5]
                 cmd                 [*]  cmd        % else [5]
                 pred                =>   cmd        % guarded cmd: if pred then cmd [5]
                 VAR declInitList    |    cmd        % variable introduction [6]
                 cmd                 ;    cmd        % sequential composition
                 cmd EXCEPT handler                  % handle exception [9]

                 << cmd >>                           % atomic brackets [7]
                 BEGIN cmd END                       % just brackets
                 IF cmd FI                           % just brackets [5]
                 DO cmd OD                           % repeat until cmd fails [8]

invocation   ::= primary arguments                   % primary has a routine type [3]

assignment   ::= lhs          := exp                 % state := state{name -> exp} [4]
                 lhs infixOp  := exp                 % short for lhs := lhs infixOp exp
                 lhs          := invocation          % of a PROC or APROC
                 ( lhsList )  := exp                 % exp a tuple that fits lhsList
                 ( lhsList )  := invocation
lhs          ::= name                                % defined in section 4
                 lhs . id                            % record field [4]
                 lhs arguments                       % function  [4]

declInit     ::= decl                                % initially any value of the type [6]
                 id : type := exp                    % initially exp, which must fit type [6]
                 id        := exp                    % short for id: T := exp, where
                                                     %  T is the type of exp

handler      ::= exceptionSet => cmd                 % [9]. See section 4 for exceptionSet
```

The ambiguity of the command grammar is resolved by taking the command composition operations `;`, `[]`, and `[*]` to be left-associative and `EXCEPT` to be right associative, and giving `[]` and `[*]` lowest precedence, `=>` and `|` next (to the right only, since their left operand is an `exp`), `;` next, and `EXCEPT` highest precedence.

[1] The empty command and `SKIP` make no change in the state. `HAVOC` produces an arbitrary outcome from any state; if you want to specify undefined behavior when a precondition is not satisfied, write `~precondition => HAVOC`.

[2] A `RET` may only appear in a routine body, and the `exp` must fit the result type of the routine. The `exp` is omitted iff the `returns` of the routine's `signature` is empty.

[3] For `arguments` see section 5. The argument are passed by value, that is, assigned to the formals of the procedure A function body cannot invoke a `PROC` or `APROC`; together with the rule for assignments (see [7]) this ensures that it can't affect the state. An atomic command can invoke an `APROC` but not a `PROC`. A command is atomic iff it is `<< cmd >>`, a subcommand of an atomic command, or one of the simple commands `SKIP`, `HAVOC`, `RET`, or `RAISE`. The type-checking rule for `invocation`s is the same as for function invocations in expressions.

[4] You can only assign to a name declared with `VAR` or in a `signature`. In an `assignment` the `exp` must fit the type of the `lhs`, or there is a fatal error. In a function body `assignments` must be to names declared in the signature or the body, to ensure that the function can't have side effects.

An assignment to a left hand side that is not a name is short for assigning a constructor to a name. In particular,

`lhs(arguments) := exp` is short for `lhs := lhs{arguments->exp}`, and
`lhs . id       := exp` is short for `lhs := lhs{id := exp}`.

These abbreviations are expanded repeatedly until `lhs` is a `name`.

In an assignment the right hand side may be an `invocation` (of a procedure) as well as an ordinary expression (which can only invoke a function). The meaning of `lhs := exp` or `lhs := invocation` is to first evaluate the `exp` or do the `invocation` and assign the result to a temporary variable `v`, and then do `lhs := v`. Thus the assignment command is not atomic unless it is inside `<<...>>`.

If the left hand side of an `assignment` is a `(lhsList)`, the `exp` must be a tuple of the same length, and each component must fit the type of the corresponding `lhs`. Note that you cannot write a tuple constructor that contains procedure invocations.

[5] A guarded command fails if the result of `pred` is undefined or `false`. It is equivalent to `cmd` if the result of `pred` is `true`. A `pred` is just a Boolean `exp`; see section 4.

`S1 [] S2` chooses one of the `Si` to execute. It chooses one that doesn't fail. Usually `S1` and `S2` will be guarded. For example,
`x=1 => y:=0 [] x> 1 => y:=1` sets `y` to `0` if `x=1`, to `1` if `x>1`, and has no outcome if `x<1`. But `x=1 => y:=0 [] x>=1 => y:=1` might set `y` to `0` or `1` if `x=1`.

`S1 [*] S2` is the same as `S1` unless `S1` fails, in which case it's the same as `S2`.

`IF ... FI` are just command brackets, but it often makes the program clearer to put them around a sequence of guarded commands, thus:

```
IF   x < 0 => y := 3
[]   x = 0 => y := 4
[*]           y := 5
FI
```

[6] In a `VAR` the unadorned form of `declInit` initializes a new variable to an arbitrary value of the declared type. The `:=` form initializes a new variable to `exp`. Precisely,

```
VAR id: T := exp | c
```

is equivalent to

```
VAR id: T | id := exp; c
```

The `exp` could also be a procedure invocation, as in an `assignment`.

Several `declInit`s after `VAR` is short for nested `VAR`s. Precisely,

```
VAR declInit ,     declInitList  | cmd
```

is short for

```
VAR declInit | VAR declInitList | cmd
```

This is unlike a module, where all the names are introduced in parallel.

[7] In an atomic command the atomic brackets can be used for grouping instead of `BEGIN ... END`; since the command can't be any more atomic, they have no other meaning in this context.

[8] Execute `cmd` repeatedly until it fails. If `cmd` never fails, the result is a looping exception that doesn't have a name and therefore can't be handled. Note that this is *not* the same as failure.

[9] Exception handling is as in Clu, but a bit simplified. Exceptions are named by literal strings (which are written without the enclosing quotes). A module can also declare an identifier that denotes a set of exceptions. A command can have an attached exception `handler`, which gets to look at any exceptions produced in the command (by `RAISE` or by an invocation) and not handled closer to the point of origin. If an exception is not handled in the body of a routine, it is raised by the routine's invocation.

An exception `ex` must be in the `RAISES` set of a routine `r` if either `RAISE ex` or an invocation of a routine with `ex` in its `RAISES` set occurs in the body of `r` outside the scope of a handler for `ex`.

[10] `CRASH` stops the execution of any current invocations in the module other than the one that executes the `CRASH`, and discards their local state. The same thing happens to any invocations outside the module from within it. After `CRASH`, no procedure in the module can be invoked from outside until the routine that invokes it returns. `CRASH` is meant to be invoked from within a special `Crash` procedure in the module that models the effects of a failure.

7. Modules

A program is some global declarations plus a set of modules. Each module contains variable, routine, exception, and type declarations.

Module definitions can be parameterized with `mformals` after the module `id`, and a parameterized module can be instantiated. Instantiation is like macro expansion: the formal parameters are replaced by the arguments throughout the body to yield the expanded body. The parameters must be types, and the body must type-check without any assumptions about the argument that replaces a formal other than the presence of a `WITH` clause that contains all the methods mentioned in the formal parameter list (that is, formals are treated as distinct from all other types).

Each module is a separate scope, and there is also a `Global` scope for the identifiers declared at the top level of the `program`. An identifier `id` declared at the top level of a non-parameterized module `m` is short for `m.id` when it occurs in `m`. If it appears in the `exports`, it can be denoted by `m.id` anywhere. When an identifier `id` that is declared globally occurs anywhere, it is short for `Global.id`. `Global` cannot be used as a module `id`.

An exported `id` must be declared in the module. If an exported `id` has a `WITH` clause, it must be declared in the module as a type with at least those methods, and only those methods are accessible outside the module; if there is no `WITH` clause, all its methods and constructors are accessible. This is Spec's version of data abstraction.


```
program       ::= toplevel* module* END

module        ::= modclass id mformals exports = body END id

modclass      ::= MODULE
                  CLASS                             % [4]

exports       ::= EXPORT exportList
export        ::= id
                  id WITH {methodList}              % see section 4 for method

mformals      ::= empty
                  [ mfpList ]
mfp           ::= id                                % module formal parameter
                  id WITH { declList }              % see section 4 for decl

body          ::= toplevel*                         % id must be the module id
                  id [ typeList ]                   % instance of parameterized module

toplevel      ::= VAR declInit*                     % declares the decl ids [1]
                  CONST declInit*                   % declares the decl ids as constant
                  routineDecl                       % declares the routine id
                  EXCEPTION exSetDecl*              % declares the exception set ids
                  TYPE typeDecl*                    % declares the type ids and any
                                                    % ids in ENUMs

routineDecl   ::= FUNC   id signature =  cmd        % function
                  APROC  id signature =<<cmd>>      % atomic procedure
                  PROC   id signature =  cmd        % non-atomic procedure
                  THREAD id signature =  cmd        % one thread for each possible
                                                    % invocation of the routine [2]
signature     ::= ( declList ) returns raises       % see section 4 for returns
                  ( )          returns raises       %  and raises

exSetDecl     ::= id = exceptionSet                 % see section 4 for exceptionSet

typeDecl      ::= id = type                         % see section 4 for type
                  id = ENUM [ idList ]              % a value is one of the id's [3]
```

[1] The "`:= exp`" in a `declInit` (defined in section 6) specifies an initial value for the variable. The `exp` is evaluated in a state in which each variable used during the evaluation has been initialized, and the result must be a normal value, not an exception. The `exp` sees all the names known in the scope, not just the ones that textually precede it, but the relation "used during evaluation of initial values" on the variables must be a partial order so that initialization makes sense. As in an `assignment`, the `exp` may be a procedure invocation as well as an ordinary expression. It's a fatal error if the `exp` is undefined or the invocation fails.

[2] Instead of being invoked by the client of the module or by another procedure, a thread is automatically invoked in parallel once for every possible value of its arguments. The thread is named by the `id` in the declaration together with the argument values. So

```
VAR sum := 0, count := 0
THREAD P(i: Int) = i IN 0 .. 9 =>
  VAR t | t := F(i); <<sum := sum + t>>; <<count := count + 1>>
```

adds up the values of `F(0) ... F(9)` in parallel. It creates a thread `P(i)` for every integer `i`; the threads `P(0), ..., P(9)` for which the guard is true invoke `F(0), ..., F(9)` in parallel and total the results in `sum`. When `count = 10` the total is complete.

A thread is the only way to get an entire program to do anything (except evaluate initializing expressions, which could have side effects), since transitions only happen as part of some thread.

[3] The `id`'s in the list are declared in the module; their type is the `ENUM` type. There are no operations on enumeration values except the ones that apply to all types: equality, assignment, and routine argument and result communication.

[4] A class is shorthand for a module that declares a convenient object type. The next few paragraphs specify the shorthand, and the last one explains the intended usage.

If the class `id` is `Obj`, the module `id` is `ObjMod`. Each variable declared in a top level `VAR` in the class becomes a field of the `ObjRec` record type in the module. The module exports only a type `Obj` that is also declared globally. `Obj` indexes a collection of state records of type `ObjRec` stored in the module's `objs` variable, which is a function `Obj->ObjRec`. `Obj`'s methods are all the names declared at top level in the class except the variables, plus the `new` method described below; the exported `Obj`'s methods are all the ones that the class exports plus `new`.

To make a class routine suitable as a method, it needs access to an `ObjRec` that holds the state of the object. It gets this access through a `self` parameter of type `Obj`, which it uses to refer to the object state `objs(self)`. To carry out this scheme, each routine in the module, unless it appears in a `WITH` clause in the class, is 'objectified' by giving it an extra `self` parameter of type `Obj`. In addition, in a routine body every occurrence of a variable `v` declared at top level in the class is replaced by `objs(self).v` in the module, and every invocation of an objectified class routine gets `self` as an extra first parameter.

The module also gets a synthesized and objectified `StdNew` procedure that adds a state record to `objs`, initializes it from the class's variable initializations (rewritten like the routine bodies), and returns its `Obj` index; this procedure becomes the `new` method of `Obj` unless the class already has a `new` routine.

A class cannot declare a `THREAD`.

The effect of this transformation is that a variable `obj` of type `Obj` behaves like an object. The state of the object is `objs(obj)`. The invocation `obj.m` or `obj.m(x)` is short for `ObjMod.m(obj)` or `ObjMod.m(obj, x)` by the usual rule for methods, and it thus invokes the method `m`; in `m`'s body each occurrence of a class variable refers to the corresponding field in `obj`'s state. `Obj.new()` returns a new and initialized `Obj` object. The following example shows how a class is transformed into a module.


```
CLASS Obj EXPORT T1, f, p, … =   MODULE ObjMod EXPORT Obj WITH {T1, f, p, new} =

TYPE T1 = … WITH {add:=AddT}     TYPE T1 = … WITH {add:=AddT}
CONST c := …                     CONST c := …

VAR  v1:T1:=ei, v2:T2:=pi(v1), … TYPE ObjRec = [v1: T1, v2: T2, …]
                                      Obj = Int WITH {T1, c, f:=f, p:=p,
                                                      AddT:=AddT, …,new:=StdNew}
                                 VAR  objs: Obj -> ObjRec := {}

FUNC f(p1: RT1, …) = … v1 …      FUNC f(self: Obj, p1: RT1, …) =
                                     … objs(self).v1 …
PROC p(p2: RT2, …) = … v2 …      PROC p(self: Obj, p2: RT2, …) =
                                     … objs(self).v2 …
FUNC AddT(t1, t2) = …            FUNC AddT(t1, t2) = …   % in T1's WITH, so not objectified
…                                …
                                 PROC StdNew(self: Obj) -> Obj =
                                   VAR obj: Obj | ~ obj IN objs.dom =>
                                     objs(obj)    := ObjRec{};
                                     objs(obj).v1 := ei;
                                     objs(obj).v2 := pi(objs(obj).v1);
                                     …;
                                     RET obj

END Obj                          END ObjMod

                                 TYPE Obj = ObjMod.Obj
```

In abstraction functions and invariants we also write `obj.n` for field `n` in `obj`'s state, that is, for `ObjMod.objs(obj).n`.

8. Scope

The declaration of an identifier is known throughout the smallest scope in which the declaration appears (redeclaration is not allowed). This section summarizes how scopes work in Spec; terms defined before section 7 have pointers to their definitions. A scope is one of

> the whole `program`, in which just the predefined (section 3), module, and globally declared identifiers are declared;
>
> a `module`;
>
> the part of a `routineDecl` or `LAMBDA` expression (section 5) after the `=`;
>
> the part of a `VAR declInit | cmd` command after the `|` (section 6);
>
> the part of a constructor or quantification after the first `|` (section 5).
>
> a record `type` or `methodDefList` (section 4);

An identifier is declared by

> a module `id`, `mfp`, or `toplevel` (for types, exception sets, `ENUM` elements, and named routines),
>
> a `decl` in a record `type` (section 4), `| constructor` or quantification (section 5), `declInit` (section 6), routine `signature`, or `WITH` clause of a `mfp`, or
>
> a `methodDef` in the `WITH` clause of a `type` (section 4).

An identifier may not be declared in a scope where it is already known. An occurrence of an identifier `id` always refers to the declaration of `id` which is known at that point, except when `id` is being declared (precedes a `:`, the `=` of a `toplevel`, the `:=` of a record constructor, or the `:=` or `BY` in a `seqGen`), or follows a dot. There are four cases for dot:

`moduleId . id` — the `id` must be exported from the basic module `moduleId`, and this expression denotes the meaning of `id` in that module.

`record . id` — the `id` must be declared as a field of the record type, and this expression denotes that field of `record`. In an `assignment`'s `lhs` see [7] in section 6 for the meaning.

`typeId . id` — the `typeId` denotes a type, `id` must be a method of this type, and this expression denotes that method.

`primary . id` — the `id` must be a method of `primary`'s type, and this expression, together with any following arguments, denotes an invocation of that method; see [2] in section 5 on expressions.

If `id` refers to an identifier declared by a `toplevel` in the current module `m`, it is short for `m.id`. If it refers to an identifier declared by a `toplevel` in the `program`, it is short for `Global.id`. Once these abbreviations have been expanded, every name in the state is either global (contains a dot and is declared in a `toplevel`), or local (does not contain a dot and is declared in some other way).

Exceptions look like identifiers, but they are actually string literals, written without the enclosing quotes for convenience. Therefore they do not have scope.

9. Built-in methods

Some of the type constructors have built-in methods, among them the operators defined in the expression grammar. The built-in methods for types other than `Int` and `Bool` are defined below. Note that these are not complete definitions of the types; they do not include the constructors.

Sets

A set has methods for

computing union, intersection, and set difference (lifted from `Bool`; see note 3 in section 4), and adding or removing an element, testing for membership and subset;

choosing (deterministically) a single element from a set, or a sequence with the same members, or a maximum or minimum element, and turning a set into its characteristic predicate (the inverse is the predicate's `set` method);

composing a set with a function or relation, and converting a set into a relation from `nil` to the members of the set (the inverse of this is just the range of the relation).

We define these operations with a module that represents a set by its characteristic predicate. Precisely, `SET T` behaves as though it were `Set[T].S`, where


```
MODULE Set[T] EXPORT S =

TYPE S = Any->Bool SUCHTHAT (ALL any | s(any) ==> (any IS T))
% Defined everywhere so that type inclusion will work; see section 4.
                     WITH {"\/":=Union, "/\":=Intersection, "-":=Difference,
                           "IN":=In, "<=":=Subset, choose:=Choose, seq:=Seq,
                            pred:=Pred, rel:=Rel, id:=Id,univ:=Univ,include:=Incl,
                            perms:=Perms,fsort:=FSort,sort:=Sort,combine:=Combine,
                            fmax:=FMax, fmin:=FMin, max:=Max, min:=Min
                            "*":=ComposeF, "**":=ComposeR }

FUNC Union(s1, s2)->S        = RET (\ t | s1(t) \/  s2(t))  % s1 \/ s2
FUNC Intersection(s1, s2)->S = RET (\ t | s1(t) /\  s2(t))  % s1 /\ s2
FUNC Difference(s1, s2)->S   = RET (\ t | s1(t) /\ ~s2(t))  % s1 - s2
FUNC In(s, t)->Bool          = RET s(t)                     % t IN s
FUNC Subset(s1, s2)->Bool    = RET (ALL t| s1(t) ==> s2(t)) % s1 <= s2
FUNC Size(s)->Int            =                              % s.size
    VAR t | s(t) => RET Size(s-{t}) + 1 [*] RET 0
FUNC Choose(s)->T            = VAR t | s(t) => RET t        % s.choose
% Not really, since VAR makes a non-deterministic choice,
% but choose makes a deterministic one. It is undefined if s is empty.
FUNC Seq(s)->SEQ T           =                              % s.seq
% Defined only for finite sets. Note that Seq chooses a sequence deterministically.
    RET {q: SEQ T | q.rng = s /\ q.size = s.size}.choose

FUNC Pred(s)->(T->Bool)      = RET s                        % s.pred
% s.pred is just s. We define pred for symmetry with seq, set, etc.
FUNC Rel(s)->(Bool->>T)      = s.pred.inv
FUNC Id(s)->(T->>T)          = RET {t :IN s || (t, t)}.pred.pToR
FUNC Univ(s)->(T->>T)        = s.rel.inv * s.rel
FUNC Incl(s)->(SET T->>T)    = (\ st: SET T, t | t IN (st /\ s)).pToR

FUNC Perms(s)->SET SEQ T          = RET s.seq.perms         % s.perms
FUNC FSort(s, f: (T,T)->Bool)->S = RET s.seq.fsort(f)      % s.fsort(f); f is compare
FUNC Sort(s)->S                   = RET s.seq.sort          % s.sort; only if T has <=
FUNC Combine(s, f: (T,T)->T)->T  = RET s.seq.combine(f)    % useful if f is commutative
FUNC FMax(s, f: (T,T)->Bool)->T  = RET s.fsort(f).last     % s.fmax(f); a max under f
FUNC FMin(s, f: (T,T)->Bool)->T  = RET s.fsort(f).head     % s.fmin(f); a min under f
FUNC Max(s)->T                    = RET s.fmax(T."<=")      % s.max; only if T has <=
FUNC Min(s)->T                    = RET s.fmin(T."<=")      % s.min; only if T has <=
% Note that these functions are undefined if s is empty. If there are extremal elements not distinguished by f or "<=",
% they make an arbitrary deterministic choice. To get all the choices, use T.f.rel.leaves.
% Note that this is not the same as /\ : s, unless s is totally ordered.
FUNC ComposeF(s, f: T->U)->SET U = RET {t :IN s || f(t)}    % s * f; image of s under f
% ComposeF like sequences, pointwise on the elements. ComposeF(s, f) = ComposeR(s, f.rel)
FUNC ComposeR(s, r:T->>U)->SET U = RET (s.rel * r).rng     % s ** r; image of s under r
% ComposeR is relational composition: anything you can get to by r, starting with a member of s.
% We could have written it explicitly: {t :IN s, u | r(t, u) || u}, or as \/ : (s * r.setF).

END Set
```

There are constructors `{}` for the empty set, `{e1, e2, ...}` for a set with specific elements, and `{declList | pred || exp}` for a set whose elements satisfy a predicate. These constructors are described in [6] and [10] of section 5. Note that `{t | p}.pred = (\ t | p)`, and similarly `(\ t | p).set = {t | p}`. A method on `T` is lifted to a method on `S`, unless the name conflicts with one of `S`'s methods, exactly like lifting on `S.rel`; see note 3 in section 4.

Functions

The function types `T->U` and `T->U RAISES XS` have methods for

composition, overlay, inverse, and restriction;

testing whether a function is defined at an argument and whether it produces a normal (non-exceptional) result at an argument, and for the domain and range;

converting a function to a relation (the inverse is the relation's `func` method) or a function that produces a set to a relation with each element of the set (`setRel`; the inverse is the relation's `setF` method).

In other words, they behave as though they were `Function[T, U].F`, where (making allowances for the fact that `XS` and `V` are pulled out of thin air):

```
MODULE Function[T, U] EXPORT F =

TYPE F = T->U RAISES XS WITH {"*":=Compose, "+":=Overlay,
                              inv:=Inverse, restrict:=Restrict,
                              "!":=Defined, "!!":=Normal,
                              dom:=Domain, rng:=Range, rel:=Rel, setRel:=SetRel}
     R = (T, U) -> Bool

FUNC Compose(f, g: U -> V) -> (T -> V) = RET (\ t | g(f(t)))
% Note that the order of the arguments is reversed from the usual mathematical convention.

FUNC Overlay(f1, f2) -> F = RET (\ t | (f2!t => f2(t) [*] f1(t)))
% (f1 + f2) is f2(x) if that is defined, otherwise f1(x)

FUNC Inverse(f) -> (U -> T) = RET f.rel.inv.func
FUNC Restrict(f, s: SET T) -> F = (s.id * f).func

FUNC Defined(f, t)->Bool =
    IF f(t)=f(t) => RET true [*] RET false FI EXCEPT XS => RET true

FUNC Normal(f, t)->Bool = t IN f.dom

FUNC Domain(f) -> SET T = f.rel.dom
FUNC Range (f) -> SET U = f.rel.rng

FUNC Rel(f) -> R = RET (\ t, u | f(t) = u).pToR
FUNC SetRel(f) -> ((T, V)->Bool) = RET (\ t, v | (f!t ==> v IN f(t) [*] false) )
% if U = SET V, f.setRel relates each t in f.dom to each element of f(t).

END Function
```

Note that there are constructors `{}` for the function undefined everywhere, `T{* -> result}` for a function of type `T` whose value is `result` everywhere, and `f{exp -> result}` for a function which is the same as `f` except at `exp`, where its value is `result`. These constructors are described in [6] and [8] of section 5. There are also lambda constructors for defining a function by a computation, described in [9] of section 5. A method on `U` is lifted to a method on `F`, unless the name conflicts with a method of `F`; see note 3 in section 4.

Functions declared with more than one argument take a single argument that is a tuple. So `f(x: Int)` takes an `Int`, but `f(x: Int, y: Int)` takes a tuple of type `(Int, Int)`. This convention keeps the tuples in the background as much as possible. The normal syntax for calling a function is `f(x, y)`, which constructs the tuple `(x, y)` and passes it to `f`. However, `f(x)` is treated differently, since it passes `x` to `f`, rather than the singleton tuple `{x}`. If you have a tuple `t` in hand, you can pass it to `f` by writing `f$t` without having to worry about the singleton case; if `f` takes only one argument, then `t` must be a singleton tuple and `f$t` will pass `t(0)` to `f`. Thus `f$(x, y)` is the same as `f(x, y)` and `f${x}` is the same as `f(x)`.

A function declared with names for the arguments, such as

```
    (\ i: Int, s: String | i + StringToInt(x))
```

has a type that ignores the names, `(Int, String)->Int`. However, it also has a method `argNames` that returns the sequence of argument names, `{"i", "s"}` in the example, just like a record. This makes it possible to match up arguments by name.

A total function `T->Bool` is a predicate and has an additional method to compute the set of `T`'s that satisfy the predicate (the inverse is the set's `pred` method). In other words, a predicate behaves as though it were `Predicate[T].P`, where

```
MODULE Predicate[T] EXPORT P =

TYPE P = T -> Bool WITH {set:=Set, pToR:=PToR}
FUNC Set(p) -> SET T = RET {t | p(t)}
END Predicate
```

A predicate with `T = (U, V)` defines a relation `U ->> V` by

```
FUNC PToR(p: (U, V)->Bool) -> (U ->> V) = RET (\u | {v | p(u, v)}).setRel
```

It has additional methods to turn it into a function `U -> V` or a function `U -> SET V`, and to get its domain and range, invert it or compose it (overriding the methods for a function). In other words, it behaves as though it were `Relation[U, V].R`, where (allowing for the fact that `W` is pulled out of thin air in `Compose`):

```
MODULE Relation[U, V] EXPORT R =

TYPE R = (U, V) -> Bool WITH {pred:=Pred, set:=R.rng, restrict:=Restrict,
                              fun:=Fun, setF:=SetFunc, dom:=Domain, rng :=Range,
                              inv:=Inverse, "*":=Compose}

FUNC Pred(r) -> ((U,V)->Bool) = RET r(u, v)
FUNC Restrict(r, s) -> R = RET s.id * r

FUNC Fun(r) -> (U -> V) =                    % defined at u iff r relates u to a single
    RET (\ u | (r.setF(u).size = 1 => r.setF(u).choose))
FUNC SetFunc(r) -> (U -> SET V) = RET (\ u | {v | r(u, v)})
% SetFunc(r) is defined everywhere, returning the set of V's related to u.

FUNC Domain(r) -> SET U = RET {u, v | r(u, v) || u}
FUNC Range (r) -> SET V = RET {u, v | r(u, v) || v}

FUNC Inverse(r) -> ((V, U) -> Bool) = RET (\ v, u | r(u, v))
FUNC Compose(r: R, s: (V, W)->Bool) -> (U, W)->Bool = % r * s
    RET (\ u, w | (EXISTS v | r(u, v) /\ s(v, w)) )

END Relation
```

A method on `V` is lifted to a method on `R`, unless there's a name conflict; see note 3 in section 4.

A relation with `U = V` is a graph and has additional methods to yield the sequences of `U`'s that are paths in the graph, and to compute the transitive closure and its restriction to exit nodes. In other words, it behaves as though it were `Graph[U].G`, where

MODULE Graph[T] EXPORT G =

```
TYPE G = T ->> T WITH {paths:=Paths, closure:=Closure, leaves:=Leaves }
     P = SEQ T

FUNC Paths(g) -> SET P = RET {p | (ALL i :IN p.dom - {0} | (g.pred)(p(i-1), p(i))
% Any p of size <= 1 is a path by this definition.
FUNC Closure(g) -> G = RET (\ t1, t2 |
    (EXISTS p | p.size > 1 /\ p.head = t1 /\ p.last = t2 /\ p IN g.paths ))
FUNC Leaves(g) -> G = RET g.closure * (g.rng - g.dom).id

END Graph
```

Records and tuples

A record is a function from the string names of its fields to the field values, and an *n*-tuple is a function from `0..n-1` to the field values. There is special syntax for declaring records and tuples, and for reading and writing record fields:

`[f: T, g: U]` declares a record with fields `f` and `g` of types `T` and `U`. It is short for

```
String->Any WITH { fields:=(\r: String->Any | (SEQ String){"f", "g"}) }
          SUCHTHAT   this.dom >= {"f", "g"}
                     /\ this("f") IS T /\ this("g") IS U
```

Note the `fields` method, which gives the sequence of field names `{"f", "g"}`.

`(T, U)` declares a tuple with fields of types `T` and `U`. It is short for

```
Int->Any WITH { fields:=(\r: nt->Any | 0..1) }
          SUCHTHAT   this.dom >= 0..1
                     /\ this(0) IS T /\ this(1) IS U
```

Note the `fields` method, which gives the sequence of field names `0..1`.

`r.f` is short for `r("f")`, and `r.f := e` is short for `r := r{"f"->e}`.

There is no special syntax for tuple fields, since you can just write `t(2)` and `t(2) := e` to read and write the third field, for example (remember that fields are numbered from 0).

Thus to convert a record `r` into a tuple, write `r.fields * r`, and to convert a tuple `t` into a record, write `r.fields.inv * t`.

There is also special syntax for constructing record and tuple values, illustrated in the following example. Given the type declaration

```
TYPE Entry = [salary: Int, birthdate: String]
```

we can write a record value

```
Entry{salary := 23000, birthdate := "January 3, 1955"}
```

which is short for the function constructor

```
Entry{"salary" -> 23000, "birthdate" -> "January 3, 1955"}.
```

The constructor `(`

```
23000, "January 3, 1955")
```

yields a tuple of type `(Int, String)`. It is short for

```
{0 -> 23000, 1 -> "January 3, 1955"}
```

This doesn't work for a singleton tuple, since `(x)` has the same value as `x`. However, the sequence constructor `{x}` will do for constructing a singleton tuple, since a singleton `SEQ T` has the type `(T)`.

Sequences

A function is called a sequence if its domain is a finite set of consecutive `Int`'s starting at 0, that is, if it has type

```
Q = Int -> T SUCHTHAT (\ q | (EXISTS size: Int | q.dom = (0 .. size-1).rng))
```

We denote this type (with the methods defined below) by `SEQ T`. A sequence inherits the methods of the function (though it overrides `+`), and it also has methods for

- `head`, `tail`, `last`, `reml`, `addh`, `addl`: detaching or attaching the first or last element,
- `seg`, `sub`: extracting a segment of a sequence,
- `+`, `size`: concatenating two sequences, or finding the size,
- `fill`: making a sequence with all elements the same,
- `zip` or `||`: making a pair of sequences into a sequence of pairs
- `<=`, `<<=`: testing for prefix or sub-sequence (not necessarily contiguous),
- `**`: composing with a relation (`SEQ T` inherits composing with a function),
- lexical comparison, permuting, and sorting,
- `iterate`, `combine`: iterating a function over each prefix of a sequence, or the whole sequence
- treating a sequence as a multiset, with operations to:
  - count the number of times an element appears, test membership and multiset equality, take differences, and remove an element (`"+"` or `"\/"` is union and `addl` adds an element).

All these operations are undefined if they use out-of-range subscripts, except that a sub-sequence is always defined regardless of the subscripts, by taking the largest number of elements allowed by the size of the sequence.

We define the sequence methods with a module. Precisely, `SEQ T` is `Sequence[T].Q`, where:

MODULE Sequence[T] EXPORTS Q =

```
TYPE I         =  Int
     Q         =  (I -> T) SUCHTHAT q.dom = (0 .. q.size-1).rng
                  WITH { size:=Size, seg:=Seg, sub:=Sub, "+":=Concatenate,
                         head:=Head, tail:=Tail, addh:=AddHead, remh:=Tail,
                         last:=Last, reml:=RemoveLast, addl:=AddLast,
                         fill:=Fill, zip:=Zip, "||":=Zip,
                         "<=":=Prefix, "<<=":=SubSeq,
                         "**":=ComposeR, lexLE:=LexLE, perms:=Perms,
                         fsorter:=FSorter, fsort:=FSort, sort:=Sort,
                         iterate:=Iterate, combine:=Combine,

                  % These methods treat a sequence as a multiset (or bag).
                         count:=Count, "IN":=In, "==":=EqElem,
                         "\/":=Concatenate, "-":=Diff, set:=Q.rng }

FUNC Size(q)-> Int  = RET q.dom.size

FUNC Sub(q, i1, i2) -> Q =
% q.sub(i1, i2); yields {q(i1),...,q(i2)}, or a shorter sequence if i1 < 0 or i2 >= q.size
    RET ({0, i1}.max .. {i2, q.size-1}.min) * q

FUNC Seg(q, i, n: I) -> Q = RET q.sub(i, i+n-1)        % q.seg(i,n); n T's from q(i)

FUNC Concatenate(q1, q2) -> Q = VAR q |                % q1 + q2
    q.sub(0, q1.size-1) = q1 /\ q.sub(q1.size, q.size-1) = q2 => RET q

FUNC Head(q) -> T = RET q(0)                           % q.head; first element
```

```
FUNC Tail(q) -> Q =                                    % q.tail;  all but first
    q.size > 0 => RET q.sub(1, q.size-1)
FUNC AddHead(q, t) -> Q = RET {t} + q                  % q.addh(t)

FUNC Last(q) -> T = RET q(q.size-1)                    % q.last; last element
FUNC RemoveLast(q) -> Q =                              % q.reml; all but last
    q # {} => RET q.sub(0, q.size-2)
FUNC AddLast(q, t) -> Q = RET q + {t}                  % q.addl(t)

FUNC Fill(t, n: I) -> Q = RET {i :IN 0 .. n-1 || t}    % yields n copies of t

FUNC Zip(q, qU: SEQ U) -> SEQ (T, U) =                 % size is the min
    RET (\ i | (i IN (q.dom /\ qU.dom) => (q(i), qU(i))))

FUNC Prefix(q1, q2) -> Bool =                          % q1 <= q2
    RET (EXISTS q | q1 + q = q2)

FUNC SubSeq(q1, q2) -> Bool =                          % q1 <<= q2
% Are q1's elements in q2 in the same order, not necessarily contiguously.
    RET (EXISTS p: SET Int | p <= q2.dom /\ q1 = p.seq.sort * q2)

FUNC ComposeR(q, r: (T, U)->Bool) -> SEQ U =           % q ** r
% Elements related to nothing are dropped. If an element is related to several things, they appear in arbitrary order.
    RET + : (q * r.setF * (\s: SET U | s.seq))

FUNC LexLE(q1, q2, f: (T,T)->Bool) -> Bool =           % q1.lexLE(q2, f);  f is <=
% Is q1 lexically less than or equal to q2. True if q1 is a prefix of q2,
% or the first element in which q1 differs from q2 is less.
    RET    q1 <= q2
       \/ (EXISTS i :IN q1.dom /\ q2.dom |   q1.sub(0, i-1) = q2.sub(0, i-1)
                                          /\ q1(i) # q2(i)) /\ f(q1(i), q2(i))

FUNC Perms(q)->SET Q =                                 % q.perms
    RET {q' | (ALL t | q.count(t) = q'.count(t))}

FUNC FSorter(q, f: (T,T)->Bool)->SEQ Int =             % q.fsorter(f); f is <=
% The permutation that sorts q stably. Note: can't use min to define this, since min is defined using sort.
    VAR ps := {p :IN q.dom.perms                       % all perms that sort q
             | (ALL i :IN (q.dom - {0}) | f((p*q)(i-1), (p*q)(i))) } |
      VAR p0 :IN ps |                                  % the one that reorders the least
         (ALL p :IN ps | p0.lexLE(p, Int."<=")) => RET p0

FUNC FSort(q, f: (T,T)->Bool) -> Q =                   % q.fsort(f); f is <=  for the sort
    RET q.fsorter(f) * q
FUNC Sort(q)->Q = RET q.fsort(T."<=")                  % q.sort; only if T has <=

FUNC Iterate(q, f: (T,T)->T) -> Q =                    % q.iterate(f)
% Yields qr = {q(0), qr(0) + q(1), qr(1) + q(2), ...), where t1 + t2 is f(t1, t2)
    RET {qr: Q |   qr.size=q.size /\ qr(0) = q(0)
               /\ (ALL i IN q.dom-{0} | qr(i) = f(qr(i-1),q(i)))}.one
FUNC Combine(q, f: T,T)->T) -> T = RET q.iterate(f).last
% Yields q(0) + q(1) + ..., where t1 + t2 is f(t1, t2)

FUNC Count(q, t)->Int = RET {t' :IN q | t' = t}.size  % q.count(t)
FUNC In(t, q)->Bool = RET (q.count(t) # 0)             % t IN q
FUNC EqElem(q1, q2) -> Bool = RET q1 IN q2.perms       % q1 == q2; equal as multisets
FUNC Diff(q1, q2) -> Q =                               % q1 - q2
    RET {q | (ALL t | q.count(t) = {q1.count(t) - q2.count(t), 0}.max)}.choose

END Sequence
```


A sequence is a special case of a tuple, in which all the elements have the same type.

`Int` has a method `..` for making sequences: `i .. j = {i, i+1, ..., j-1, j}`. If `j < i`, `i .. j = {}`. You can also write `i .. j` as `{k := i BY k + 1 WHILE k <= j}`; see [11] in section 5. `Int` also has a `seq` method: `i.seq = 0 .. i-1`.

There is a constructor `{e1, e2, ...}` for a sequence with specific elements and a constructor `{}` for the empty sequence. There is also a constructor `q{e1 -> e2}`, which is equal to `q` except at `e1` (and undefined if `e1` is out of range). For the constructors see [6] and [8] of section 5. To generate a sequence there are constructors `{x :IN q | pred || exp}` and `{x := e1 BY e2 WHILE pred1 | pred2 || exp}`. For these see [11] of section 5.

To map each element `t` of `q` to `f(t)` use function composition `q * f`. Thus if `q: SEQ Int`, `q * (\ i: Int | i*i)` yields a sequence of squares. You can also write this `{i :IN q || i*i}`.

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5. Examples of Specs and Code

This handout is a supplement for the first two lectures. It contains several example specs and code, all written using Spec.

Section 1 contains a spec for sorting a sequence. Section 2 contains two specs and one code for searching for an element in a sequence. Section 3 contains specs for a read/write memory. Sections 4 and 5 contain code for a read/write memory based on caching and hashing, respectively. Finally, Section 6 contains code based on replicated copies.

1. Sorting

The following spec describes the behavior required of a program that sorts sets of some type `T` with a `"<="` comparison method. We do not assume that `"<="` is antisymmetric; in other words, we can have `t1 <= t2` and `t2 <= t1` without having `t1 = t2`, so that `"<="` is not enough to distinguish values of `T`. For instance, `T` might be the record type `[name:String, salary: Int]` with `"<="` comparison of the `salary` field. Several `T`'s can have different `name`s but the same `salary`.

```
TYPE S = SET T
     Q = SEQ T

APROC Sort(s) -> Q = <<
    VAR q | (ALL t | s.count(t) = q.count(t)) /\ Sorted(q) => RET q >>
```

This spec uses the auxiliary function `Sorted`, defined as follows.

```
FUNC Sorted(q) -> Bool = RET (ALL i :IN q.dom - {0} |  q(i-1) <= q(i))
```

If we made `Sort` a `FUNC` rather than a `PROC`, what would be wrong?[1] What could we change to make it a `FUNC`?

We could have written this more concisely as

```
APROC Sort(s) -> Q =
    << VAR q :IN a.perms | Sorted(q) => RET q >>
```

using the `perms` method for sets that returns a set of sequences that contains all the possible permutations of the set.

[1] Hint: a `FUNC` can't have side effects and must be deterministic (return the same value for the same arguments).