Random Sampling and Approximation of MAX-CSP Problems

Noga Alon
Department of Mathematics
Tel Aviv University
Tel Aviv*

W. Fernandez de la Vega
CNRS
Universite Paris Sud
Orsay †

Ravi Kannan
Department of Computer Science
Yale University ‡

Marek Karpinski
Department of Computer Science
University of Bonn §

Abstract

In a MAX-rCSP problem with variables $\{x_1, x_2, \dots x_n\}$, we are given Boolean functions $f_1, f_2, \dots f_m$ each involving r of the n variables and are to find the maximum number of these functions that can be made true by a truth assignment to the variables. We show that for r fixed, there is an integer $q \in O(\log(1/\epsilon)/\epsilon^4)$ such that if we choose q variables (uniformly) at random, the answer to the sub-problem induced on the chosen variables is, with high probability, within an additive error of ϵq^r of $\frac{q^r}{n^r}$ times the answer to the original $n-$ variable problem. The previous best result for the case of $r = 2$ (which includes many Graph problems) was that there is an algorithm which given the induced sub-problem on $q = O(1/\epsilon^5)$ variables, can find an approximation to the answer to the whole problem within additive error ϵn^2. For $r \geq 3$, the conference version of this paper [4] and independently [1] give the first results with sample complexity q dependent only polynomially upon $1/\epsilon$. [1] has a sample complexity q of $O(1/\epsilon^7)$. They (as also the earlier papers) however do not directly prove any relation between the answer to the sub-problem and the whole problem as we do here.

*Supported in part by a USA-Israeli BSF grant, by the Israel Science Foundation and by the Hermann Minkowski Minerva Center for Geometry at Tel Aviv University. E-mail: noga@math.tau.ac.il

†Supported in part by PROCOPE Project and by IST grant 1496 (RAND-APX). Part of this work was done while visiting Department of Computer Science, Yale University. E-mail: lalo@lri.fr.

‡Supported in part by NSF Grant CCR-9820850. E-mail: kannan@cs.yale.edu.

§Supported in part by DFG grants, Max-Planck Research Prize, and by IST grant 14036 (RAND-APX). Part of this work was done while visiting Department of Computer Science, Yale University. E-mail: marek@cs.uni-bonn.de.

Our method also differs from other results in that it is Linear Algebraic, rather than combinatorial in nature.

1 Introduction

Suppose r is a fixed integer. In the MAX-rSAT problem, we are given a Conjunctive Normal Form Boolean formula on n variables, with each clause being the OR of precisely r literals. The objective is to maximize the number of clauses satisfied by an assignment to the n variables. The exact problem is NP-hard for each fixed $r \geq 2$. A special case of our result is that for any $\epsilon > 0$, there is a positive integer $q \in O(\log(1/\epsilon)/\epsilon^4)$ such that if we pick at random a subset of q variables (among the n) and solve the "induced" problem on the q variables (maximize the number of clauses satisfied among those containing only those variables and their negations), then the answer multiplied by n^r/q^r is, with high probability, within an additive factor ϵn^r of the optimal answer for the n variable problem. The q needed here will be called the "sample complexity" of the problem for obvious reasons.

In fact, we show the same result for all MAX-rCSP problems. (MAX-rCSP problems, also called MAX-rFUNCTION-SAT, are equivalent to MAX-SNP [6]). Recall that the input to a MAX-rCSP problem (for r fixed) consists of a set F of m distinct Boolean functions $f_1, f_2, \ldots f_m$ of n Boolean variables $x_1, x_2, \ldots x_n$, where each f_i is a function of only r of the n variables. The answer Max(F) is the maximum number of functions which can be simultaneously set to 1 by a truth assignment to the variables. For a subset Q of the variables, we let F^Q denote the subset of F which are functions of only the variables in Q (and their negations).

Theorem 1. (Main Theorem) *Let r, n be positive integers, with r fixed. Suppose ϵ is a positive real. There exists a positive integer $q \in O(\log(1/\epsilon)/\epsilon^4)$ such that for any F (as above), if Q is a random subset of $\{x_1, x_2, \ldots x_n\}$ of cardinality q, then with probability at least 9/10, we have*

$$|\frac{n^r}{q^r} Max(F^Q) - Max(F)| \leq \epsilon n^r.$$

We note that while, normally, sampling is used to estimate certain specific quantities, here the result actually says that the sample estimates an optimal solution value well.

It is worth noting that one half of the Theorem - namely the assertion that

$$\frac{n^r}{q^r}\text{Max}(F^Q) - \text{Max}(F) \geq -\epsilon n^r$$

is relatively easy to prove. Indeed, if, the assignment of truth values to $x_1, x_2, \ldots x_n$ achieving Max(F) sets to 1 a set S of functions among $f_1, f_2, \ldots f_m$, one can show that a sufficient number of functions in S are in F^Q just from the fact that Q is random. This then says that the same assignment restricted to Q sets to 1 a sufficient number of functions. So, a good solution to the whole problem provides also good solutions to random induced sub-problems. (We will see this argument in Section 7.) The other half -

$$\frac{n^r}{q^r}\mathrm{Max}(F^Q) - \mathrm{Max}(F) \leq \epsilon n^r$$

is much harder. Intuitively, for proving this part, we have to show that if there is no good solution to the whole problem, then also, there are no good solutions to random induced sub-problems.

The MAX-rSAT and other MAX-rCSP problems all admit fixed factor relative approximation algorithms which run in polynomial time. For some MAX-SNP problems, there have been major breakthroughs in achieving better factors using semi-definite programming and other techniques [12]. More relevant to our paper is the line of work started with the paper of Arora, Karger and Karpinski [6] which introduced the technique of smooth programs, and designed the first polynomial time algorithms for solving MAX-SNP problems (of arity r) to within additive error guarantee ϵn^r, for each fixed $\epsilon > 0$. Frieze and Kannan [10] proved an efficient version of Szémeredi's Regularity Lemma and used it to get a uniform framework to solve all MAX-SNP and some other problems in polynomial time with the same additive error. In [11], they introduced a new way of approximating matrices and more generally r-dimensional arrays, called the "cut-decomposition" and using those, proved a result somewhat similar to the main result here (for each fixed r), but with two important differences - (i) the sample complexity was exponential in $1/\epsilon$ and (ii) their result did not relate the optimal solution value of the whole problem to the optimal solution of the random sub-problems in their original setting; instead it related it to a complicated computational quantity associated with the random sub-problem. We will make central use of cut-decompositions in this paper.

For the special case of $r = 2$, Goldreich, Goldwasser and Ron [13] designed algorithms, where the sample complexity was polynomial in $1/\epsilon$; indeed, by exploiting the special structure of individual problems like the MAX-CUT problem they improved the polynomial dependence. Their results relate the optimal solution value of the whole problem to a complicated function of the random sub-problems like [10], but, as a corollary, they also obtain a less efficient version relating it to the optimal solution of the random sub-problems. See also [10], [8] and [5] for higher dimensional cases, or for cases in which our only objective is to decide if we can satisfy almost all constraints. Our new method here is more uniform and general.

Our result is derived by general arguments about approximating multi- (and 2-) dimensional arrays by some simple arrays and then using Linear Programming arguments. Unlike previous papers, we do not use problem-specific arguments which dwell into the special structure of individual problems. The MAX-CUT problem (a special MAX-2CSP problem) has received much attention in this context. Indeed, independently of the papers so far cited, Fernandez de la Vega [9] developed a different algorithm for this problem which within polynomial time, produced a solution with additive error ϵn^2. [13] used the special structure of the problem to derive an algorithm with sample complexity $O(1/\epsilon^5)$ (best known upto now).

An earlier version of this paper proving the main theorem with $q = O(\log(1/\epsilon)/\epsilon^4)$ for the case $r = 2$ and $q = O(\log(1/\epsilon)/\epsilon^{12})$ for the case of general r appeared in [4]. Independently of our work, Anderson and Engebretsen [1] (see also [2]) have obtained a constant time approximation algorithm for MAX-rCSP. They state their results within the query model of [13] and their algorithm makes $O(\log^2(1/\epsilon)/\epsilon^7)$ queries for accuracy ϵn^r.

Here is an outline of our method : In Section 2, we represent MAX-rCSP problems by r-dimensional arrays. In Section 3, following the approach of [11], we show how to approximate any r-dimensional arrays by the sum of a small number of "cut-arrays". These cut arrays are analogs of rank 1 matrices in the case of 2-dimensional arrays and so the approximation itself is an analog of approximating a 2-dimensional matrix by the sum of a small number of rank 1 matrices. As a warm-up to the main result, in Section 4, we show how to solve the MAX-CSP problem approximately by explicitly finding the cut approximation. [This is not a "constant" time algorithm.] In Section 5, we prove that cut approximation for the full array also works for a random sub-array on a random subset of $O(\log(1/\epsilon)/\epsilon^4)$ elements. This is technically perhaps the hardest part of the paper.

Then in Section 6, we show a result about Linear Programs to be used later; this is potentially of independent interest. The result says that given a Linear Program on n variables, all constrained to be between 0 and 1, if we pick (uniformly) at random a (small) subset of variables and consider the Linear Program on these variables, the optimal value of this Linear Program gives us a good "estimate" of the optimal value of the whole Linear Program. The proof is relatively simple; it uses Linear Programming Duality crucially.

Finally, Section 7 puts it all together - we argue as follows : If the n variable MAX-CSP problem has optimal solution with value αn^r, it is easy to see that the induced sub-problem on q randomly chosen variables has an optimal answer of at least αq^r minus a small error by just examining the solution to the sub-problem contained in the optimal solution to the whole problem. The converse requires all the work. First we argue that a natural Linear Programming relaxation of the whole optimization problem has a maximum solution value of at most αn^r plus a small amount. Then we use the result on Linear Programs mentioned above to assert that the corresponding Linear Program induced on the randomly chosen variables also has its maximum solution value bounded above. We then use the result of Section 5 to argue that this implies that the solution value of the MAX-CSP problem induced on the chosen variable is small.

Thus, in order to approximate any problem from MAX-rCSP, it is enough to find a good approximation to the optimum of an induced random subsystem. As a consequence, our sample bound above gives, by a direct application of an approximation method of [6], a running time of $2^{\tilde{O}\left(\frac{1}{\epsilon^2}\right)}$ for approximating all MAX-rCSP problems, which is also an improvement of the previous best known results which have higher powers of $1/\epsilon$ in the exponent.

2 Max-rCSP and r-dimensional arrays

A Max-rCSP (maximum-r-Constraint Satisfaction) Problem with variables $x_1, x_2, \ldots x_n$ consists of a given set of m Boolean Functions $f_1, f_2, \ldots f_m$, all distinct, where each f_i is a Boolean function of r of the variables. r is considered fixed, whereas n goes to infinity in our asymptotic analysis. The objective is to maximize the number of Boolean functions (among $f_1, f_2, \ldots f_m$) satisfied by assignment of truth values to the variables $x_1, x_2, \ldots x_n$. Max-2CSP includes many Graph problems like the maximum cut problem, and other graph partitioning problems. Max-rCSP includes as a special case, the problem of maximizing the number of satisfied clauses in a CNF Boolean formula with r literals per clause.

The first contribution of this paper is to give a Linear Algebra based algorithm which solves the problem to relative error δ in time which grows as $2^{\tilde{O}(n^r/\delta^2 m)}O(n^r)$. Note that m/n^r is, up to a constant depending on r, the density of the problem, that is, the fraction of functions that appear in the problem among all functions of r of the variables. Therefore, this algorithm is better than the trivial 2^n algorithms as long as $m >> n^{r-1}$. In the case $r = 2$ (graphs), this requirement just says that we have a super-linear (in n) number of edges.

We may reduce a Max-rCSP problem to the problem of maximizing a polynomial of degree r over the (vertices of the) unit Boolean cube - $C = \{0,1\}^n$ as follows :

Let $V = \{1, 2, \ldots n\}$. For each $0,1$ sequence z of length r, $z = (z_1, z_2, \ldots z_r)$, we define the r-dimensional array $A^{(z)}$ on V^r where $A^{(z)}(i_1, \ldots i_r)$ is the number of functions among $\{f_1, f_2, \ldots f_m\}$ which are made true by the assignment $x_{i_1} = z_1, \ldots x_{i_r} = z_r$. (Obviously, an f_i must be a function of $x_{i_1}, x_{i_2}, \ldots x_{i_r}$ to contribute to $A^{(z)}(i_1, i_2, \ldots i_r)$.) Then the polynomial

$$P(x) = \sum_{z \in \{0,1\}^r} \sum_{i_1, i_2, \ldots i_r} A^{(z)}(i_1, i_2, \ldots i_r) \prod_{j : z_j = 1} x_{i_j} \prod_{j : z_j = 0} (1 - x_{i_j}). \tag{1}$$

over x of degree r gives us the number of satisfied clauses for the truth assignment x to the variables. [This is because, in any truth assignment, each f_i is counted at most once as being satisfied.]

We will view each $A^{(z)}$ as an r-dimensional array on V^r, i.e., $A^{(z)} : V^r \to \mathbf{R}$. [Note that 2-dimensional arrays are just matrices.] To maximize the polynomial $P(x)$ over $\{0,1\}^n$, we first approximate each $A^{(z)}$ by what we call a "cut decomposition" and then solve the corresponding maximization problem with $A^{(z)}$ replaced by its cut decomposition. We presently describe this in more detail.

For ease of notation, let $V_1, V_2, \ldots V_r$ be (not necessarily distinct) finite sets. An r-dimensional array A on $V_1, V_2, \ldots V_r$ is a function $A : V_1 \times V_2 \times \ldots V_r \longrightarrow \mathbf{R}$. [In our case, $V_i = V$ for all i.] For each $i_1 \in V_1, i_2 \in V_2, \ldots i_r \in V_r$, we call $A(i_1, i_2, \ldots i_r)$ an entry of A. We let $||A||_F$ be the square root of the sum of squares of all the entries. [This is sometimes called the Frobenius norm, hence the subscript F.]

For any $S_1 \subseteq V_1, S_2 \subseteq V_2 \dots S_r \subseteq V_r$ we let

$$A(S_1, S_2, \dots S_r) = \sum_{(i_1,i_2,\dots i_r) \in S_1 \times S_2 \times \dots S_r} A(i_1, i_2, \dots i_r).$$

Define another norm $||A||_C$ (called the cut norm) :

$$A^+ = \max_{S_1 \subseteq V_1, S_2 \subseteq V_2, \dots S_r \subseteq V_r} A(S_1, S_2, \dots S_r)$$
$$\text{and } ||A||_C = \max(A^+, (-A)^+).$$

The cut norm was defined and studied in [11].

For any $S_1, S_2, \dots S_r$, and real value d we define the *Cut Array* $C = CUT(S_1, S_2, \dots S_r; d)$

$$C(i_1, i_2, \dots i_r) = \begin{cases} d & \text{if } (i_1, i_2, \dots i_r) \in S_1 \times S_2 \dots S_r, \\ 0 & \text{otherwise.} \end{cases}$$

The real number d is called the coefficient of the cut array.

There is another way of looking at arrays which may be useful : we may view A as (the coefficient array of) a multi-linear form. To this end, let x be a vector of $|V_1|$ (real valued) variables, y a set of $|V_2|$ variables, z a set of $|V_3|$ variables etc., where the $|V_1| + |V_2| + \dots |V_r|$ variables are all considered distinct. Then we may associate A with the multi-linear form

$$\sum_{(i_1,i_2,\dots i_r) \in V_1 \times V_2 \times \dots V_r} A(i_1, i_2, \dots i_r) x_{i_1} y_{i_2} z_{i_3} \dots$$

Note that for solving the MAX-r-CSP problem, what we had was not a multi-linear form, but a polynomial; there the variables were not all distinct. Viewing the arrays $A^{(z)}$ as multi-linear forms is conceptually useful; it is not literally how we will solve the MAX$-r-$CSP problem.

By definition, $A(S_1, S_2, \dots S_r)$ is the value of the multi-linear form when we set the variable corresponding to $S_1 \cup S_2 \cup \dots S_r$ to 1 and the other variables to 0. Also, the cut array $CUT(S_1, S_2, \dots S_r; d)$ corresponds to the multi-linear form :

$$d \cdot x(S_1) y(S_2) z(S_3) \dots \qquad \text{where we use the notation} \qquad x(S) = \sum_{j \in S} x_j.$$

Thus cut arrays correspond to simple multi-linear forms which are just products of r linear forms (each of the special form $x(S)$). This representation lets us interpret the cut norm in a natural way - suppose B is another array on $V_1 \times V_2 \dots V_r$ which approximates A well in cut norm, i.e., say

$$||A - B||_C \le \Delta.$$

Then, we claim that the multi-linear forms corresponding to A and B differ by at most Δ for any setting of the variables $x, y, z \dots$ in the range - $[0, 1]$. This is because, once all variables except the x 's are fixed, we have a linear form in the x 's and so, the maximum difference between A and B is attained at a point with each x_i equal to 0 or 1. Applying this argument r times, we get that the maximum and minimum of the multi-linear form corresponding to $A - B$ are both attained at 0-1 points and so the claim follows. Thus, we have

Claim *Suppose we have arrays $B^{(z)}, z \in \{0,1\}^r$ such that $||A^{(z)} - B^{(z)}||_C \leq \Delta$ for all z, then the maximum value of the function $P(x)$ in (1) over $\{0,1\}^n$ and the maximum value of the function obtained by replacing each $A^{(z)}$ by the corresponding $B^{(z)}$ (over $\{0,1\}$) differ by at most $2^r\Delta$.*

We use one other piece of notation : for any $Q \subseteq V_2 \times V_3 \ldots V_r$, we define

$$\mathrm{Pos}(Q) = \{z \in V_1 : A(\{z\}, Q) > 0\}.$$

Note that Pos is with reference to an array A. If it is not clear from the context which array Pos is in reference to, we indicate the array as a subscript on Pos. If $Q \subset V_1 \times \ldots \times V_{i-1} \times V_{i+1} \ldots \times V_r$, then $\mathrm{Pos}(Q) \subset V_i$ is defined analogously.

If the m functions $f_1, f_2, \ldots f_m$ of our MAX-rCSP instance are all distinct, then since there are at most 2^{2^r} functions of r variables, we have for all z,

$$|A^{(z)}(i_1, i_2, \ldots i_r)| \leq 2^{2^r} \qquad ||A^{(z)}||_F^2 \leq m2^{2^r}. \tag{2}$$

3 An Explicit Algorithm for Approximation by Cut Arrays

Throughout this section, A is an array on $V_1 \times V_2 \times \ldots V_r$. We let $N = |V_1||V_2| \ldots |V_r|$.

The aim of this section is to develop an algorithm which approximates A by the sum D of a small number of cut arrays; so that $||A - D||_C$ is smaller than a certain threshold. [In other words, the multi-linear functions represented by A, D are close on the unit cube.] To this end, we first want to check whether $||A||_C$ is already smaller than the threshold. [if so, we may stop, because then the all-zero array is a good approximation.] The problem of finding $||A||_C$ is reducible to that of finding A^+ (and then $(-A)^+$.) This is NP-hard to do exactly. But, we will first describe an algorithm to find A^+ within additive error $\epsilon\sqrt{N}||A||_F$ in time $2^{O(1/\epsilon^2)}O(N)$ which will suffice for us. The algorithm is a direct consequence of the following lemma.

Lemma 2. *Let $p \geq 4r^2/(\delta^2\epsilon^2)$. For $i = 1, 2, \ldots r$, let Q_i be a random subset of $V_1 \times V_2 \times \ldots V_{i-1} \times V_{i+1} \ldots V_r$ of cardinality p.* [1] *Then with probability at least $1 - \delta$ (over the choice of $Q_1, Q_2, \ldots, Q_r$), we have :*

$$\exists Q_1' \subseteq Q_1, \exists Q_2' \subseteq Q_2, \ldots \exists Q_r' \subseteq Q_r,$$

such that

$$A(\mathrm{Pos}(Q_1'), \mathrm{Pos}(Q_2'), \ldots \mathrm{Pos}(Q_r')) \geq A^+ - \frac{\epsilon\sqrt{N}}{2}||A||_F.$$

[1] So, each of the $\binom{|V_2||V_3|\ldots|V_r|}{p}$ subsets is equally likely to be picked to be Q_1, and similarly for $Q_2, Q_3, \ldots Q_r$.

To prove the lemma, we first prove the following.

Lemma 3. *Suppose $S_1 \subseteq V_1, S_2 \subseteq V_2, \ldots S_r \subseteq V_r$ are some fixed subsets. Let p be a positive integer. Suppose Q_1 is a random subset of $V_2 \times V_3 \times \ldots V_r$ of cardinality p. Then, we have :*

$$E_{Q_1}\left(A(\mathrm{Pos}(Q_1 \cap (S_2 \times S_3 \ldots S_r)), S_2, S_3, \ldots, S_r)\right) \geq A(S_1, S_2, \ldots S_r) - \frac{\sqrt{N}}{\sqrt{p}}||A||_F.$$

Proof Let $S_2 \times S_3 \ldots \times S_r = S$. We have,

$$A(\mathrm{Pos}(Q_1 \cap S), S) = A(\mathrm{Pos}(S), S) - A(B_1, S) + A(B_2, S), \tag{3}$$

where

$$\begin{aligned} B_1 &= \{z \in V_1 :\ A(z, S) > 0 \text{ and } A(z, S \cap Q_1) < 0\}, \\ B_2 &= \{z \in V_1 :\ A(z, S) < 0 \text{ and } A(z, S \cap Q_1) > 0\}, \end{aligned}$$

Consider one fixed $z \in V_1$. Let $X_z = A(z, S \cap Q_1)$. We may write the random variable X_z as the sum $X_1 + X_2 + \ldots X_p$, where $X_1, X_2, \ldots X_p$ is a sample of size p drawn uniformly without replacement from the set of $l = |V_2| \times |V_3| \times \ldots |V_r|$ reals - $\{A(z, y)_{y \in S}\}$. For analysis, we also introduce the random variables $Y_1, Y_2, \ldots Y_p$ - a sample of size p drawn independently, each uniformly distributed over the same set of reals, but now with replacement. We have

$$E(X_1 + X_2 + \ldots X_p) = \frac{p}{l}A(z, S)$$

and

$$\begin{aligned} &\mathrm{Var}(X_1 + X_2 + \ldots X_p) \leq \mathrm{Var}(Y_1 + Y_2 + \ldots Y_p) \\ &\leq \frac{p}{l}\sum_{u \in S} A(z, u)^2 \leq \frac{p}{l}\sum_{u \in V_2 \times V_3 \times \ldots V_r} A(z, u)^2, \end{aligned}$$

where the second line is a standard inequality (for example, it follows from Theorem 4 of [14]). Hence, for any $\xi > 0$,

$$\mathbf{Pr}\left(\left|X_z - \frac{p}{l}A(z, S)\right| \geq \xi\right) \leq \frac{p\sum_{u \in V_2 \times V_3 \times \ldots V_r} A(z, u)^2}{l\xi^2} \tag{4}$$

If $z \in B_1$ then, by the definition of X_z, $X_z < 0$ and hence $X_z - (p/l)A(z, S) \leq -(p/l)A(z, S)$ and so applying (4) with $\xi = pA(z, S)/l$ we get that for each fixed z,

$$\mathbf{Pr}(z \in B_1) \leq \mathbf{Pr}(\ |X_z - (p/l)A(z, S)| > (p/l)A(z, S)\ ) \leq \frac{l\sum_{u \in V_2 \times V_3 \times \ldots V_r} A(z, u)^2}{pA(z, S)^2}.$$

$$\begin{aligned} \text{So, } \mathbf{E}\left(\sum_{z \in B_1} A(z, S)\right) &\leq \sum_{\{z \in V_1 :\ A(z,S) > 0\}} \min\left\{A(z, S), \frac{l\sum_u A(z, u)^2}{pA(z, S)}\right\} \\ &\leq \sum_{\{z \in V_1 :\ A(z,S) > 0\}} \sqrt{\frac{l\sum_{u \in V_2 \times V_3 \ldots V_r} A(z, u)^2}{p}} \end{aligned} \tag{5}$$

By an identical argument we obtain

$$\mathbf{E}\left(\sum_{z\in B_2} A(z,S)\right) \geq -\sum_{\{z\in V_1:\, A(z,S)<0\}} \sqrt{\frac{l\sum_u A(z,u)^2}{p}},$$

where u runs over $V_2 \times V_3 \times \ldots \times V_r$. Hence, (using the Cauchy-Schwartz inequality),

$$\mathbf{E}(A(\mathrm{Pos}(Q_1 \cap S), S)) \geq A(\mathrm{Pos}(S), S) - \sum_{z\in V_1} \sqrt{\frac{l\sum_u A(z,u)^2}{p}}$$

$$\geq A(\mathrm{Pos}(S), S) - \frac{\sqrt{N}}{\sqrt{p}}||A||_F,$$

completing the proof of Lemma 3.

□

By repeatedly applying the last lemma r times, we conclude that if we define $Q'_1 = Q_1 \cap (S_2 \times S_3 \ldots S_r)$, $Q'_2 = Q_2 \cap (\mathrm{Pos}(Q'_1) \times S_3 \times S_4 \ldots S_r)$, $Q'_3 = Q_3 \cap (\mathrm{Pos}(Q'_1) \times \mathrm{Pos}(Q'_2) \times S_4 \ldots S_r)$ and so on, then the expected value (over the choice of the sets Q_i) of

$$A(\mathrm{Pos}(Q'_1), \mathrm{Pos}(Q'_2), \ldots, \mathrm{Pos}(Q'_r))$$

is at least $A(S_1, S_2, \ldots, S_r) - r\frac{\sqrt{N}}{\sqrt{p}}||A||_F$.

In particular, if we let $S_1, S_2, \ldots, S_r$ be the sets that attain A^+, then $A(S_1, S_2, \ldots, S_r) - A((\mathrm{Pos}(Q'_1), \mathrm{Pos}(Q'_2), \ldots, \mathrm{Pos}(Q'_r))$ is a nonnegative random variable whose expectation is at most $r\frac{\sqrt{N}}{\sqrt{p}}||A||_F$. The assertion of Lemma 2 thus follows from Markov's Inequality.

□

Now we will apply Lemma 2 repeatedly to find an approximation of any array as a sum of cut arrays.

Theorem 4. *It is possible to find, in time $2^{O(1/\epsilon^2)}O(N)$ and with probability at least, say, $9/10$, a set of at most $4/\epsilon^2$ cut arrays whose sum, denoted D, satisfies the following inequalities :*

$$||A - D||_C \leq \epsilon\sqrt{N}||A||_F \tag{6}$$

$$||A - D||_F \leq ||A||_F \tag{7}$$

$$\text{The sum of the absolute values of the coefficients of the cut arrays } \leq \frac{2||A||_F}{\epsilon\sqrt{N}}. \tag{8}$$

This upper estimate on the number of cut arrays is tight up to the dependence on the dimension r.

Proof We are going to find cut arrays $D^{(1)}, D^{(2)}, \ldots D^{(t)}$ one by one. We start with $t = 0$. At a general stage, suppose we already have $D^{(1)}, \ldots D^{(t)}$. Let $W = A - (D^{(1)} +$

$D^{(2)} + \cdots + D^{(t)}$). We assume for induction that $||W||_F \leq ||A||_F$, which we will prove will hold at the next step.

We will use Lemma 2 with, say, $\delta = 1/2$ on W. I.e., we pick , $\log(80/\epsilon^2)$ times, random sets $Q_1, Q_2, \ldots Q_r$ of cardinality $p = O(1/\epsilon^2)$ each, try all subsets $Q'_1, Q'_2, \ldots Q'_r$ of these sets and check if for some choice of the sets Q_i, Q'_i

$$W(\mathrm{Pos}_W(Q'_1), \mathrm{Pos}_W(Q'_2), \ldots \mathrm{Pos}_W(Q'_r)) \geq \epsilon\sqrt{N}||A||_F/2. \tag{9}$$

If (9) holds for some such choice, then we let $S_1 = \mathrm{Pos}(Q'_1); S_2 = \mathrm{Pos}(Q'_2) \ldots S_r = \mathrm{Pos}(Q'_r)$ and

$$n_{t+1} = |S_1||S_2| \ldots |S_r| \tag{10}$$

$d_{t+1} = W(S_1, S_2, \ldots S_r)/n_{t+1}$ i.e., the average of the entries of W in $S_1 \times S_2 \times S_r$.

$$D^{(t+1)} = CUT(S_1, S_2, \ldots S_r; d_{t+1}), \tag{11}$$

and we go on to the next t. Noting that subtracting the cut array $D^{(t+1)}$ from W just corresponds to subtracting the average from a set of real numbers, we have:

$$||W - D^{(t+1)}||_F^2 - ||W||_F^2$$

$$= \sum_{i_1 \in S_1, i_2 \in S_2, \ldots} ((W(i_1, i_2, \ldots i_r) - d_{t+1})^2 - (W(i_1, i_2, \ldots i_r))^2) = -n_{t+1} d_{t+1}^2. \tag{12}$$

Since $|d_{t+1}|n_{t+1} \geq \epsilon\sqrt{N}||A||_F/2$, we have $|d_{t+1}|^2 n_{t+1}^2 \geq \epsilon^2 N||A||_F^2/4$ and so $|d_{t+1}|^2 n_{t+1} \geq \epsilon^2||A||_F^2/4$, proving that this process will not be repeated more than $4/\epsilon^2$ times as claimed in the Theorem.

Suppose (9) does not hold for any subsets $Q'_1, Q'_2, \ldots Q'_r$ in all the choices of the sets Q_i. If at this point in the algorithm $W^+ \geq \epsilon\sqrt{N}||A||_F$, then by Lemma 2, the probability that we fail to find any $Q'_1, \ldots Q'_r$ is at most $\epsilon^2/80$. Thus, over all steps in the algorithm the probability that this ever happens is at most $1/20$. We now repeat the process on $-W$ and if we find $Q'_1, \ldots Q'_r$ satisfying

$$-W(\mathrm{Pos}_{-W}(Q'_1), \mathrm{Pos}_{-W}(Q'_2), \ldots \mathrm{Pos}_{-W}(Q'_r)) \geq \epsilon\sqrt{N}||A|||_F/2,$$

we again define a cut matrix and proceed as above. Otherwise, if $(-W)^+ \geq \epsilon\sqrt{N}||A||_F$, then by Lemma 2, the probability that we fail to find any $Q'_1, \ldots Q'_r$ is at most $1/20$. Thus when we terminate, (6) holds with probability at least $9/10$ as claimed.

(7) holds because we see that in subtracting each cut array, we are subtracting the average of some entries from the entries, so the sum of squares only decreases.

To prove (8), we proceed as follows :

$$\sum_t d_t^2 n_t \leq ||A||_F^2 \qquad \text{from (12).}$$

As we have shown previously
$$|d_t|n_t \geq \epsilon\sqrt{N}||A||_F/2$$
and hence
$$\sum_t d_t^2 n_t \geq \frac{\epsilon^2 N||A||_F^2}{4}(\sum_t(1/n_t)).$$
Therefore,
$$\sum_t(1/n_t) \leq 4/\epsilon^2 N$$
and by Cauchy-Schwartz
$$\sum_t |d_t| \leq \left(\sum_t d_t^2 n_t\right)^{1/2} \left(\sum(1/n_t)\right)^{1/2} \leq ||A||_F \frac{2}{\epsilon\sqrt{N}}.$$
The proof of the tightness of the upper estimate is included in Section 8. □

4 Explicit Algorithm for MAX-rCSP

In this section, we prove the following Theorem :

Theorem 5. *Given an instance of a MAX-r CSP problem with n variables and m functions, each of r variables, and a $\delta > 0$, there is an algorithm running in time $2^{\tilde{O}(n^r/(\delta^2 m))}O(n^r)$ which with probability at least 9/10, outputs an assignment satisfying at least the maximum number of satisfiable functions minus δm.*

We remark that in the case of $r = 2$, this starts giving us sub-exponential algorithms as soon as the number m of edges is super-linear in n. [Clearly we may assume that $m \in \Omega(n)$; otherwise we may usually split into connected components and solve the problem on each component. Thus for all $\delta \in \Omega(1)$, our algorithm will be at least as good as the trivial 2^n algorithm.]

Throughout this section, we let
$$\epsilon = \frac{\delta\sqrt{m}}{4n^{r/2}2^{2^r}}.$$
Recall from (1) that we wish to maximize the polynomial P over $\{0,1\}^n$:
$$P(x) = \sum_{z\in\{0,1\}^r} \sum_{i_1,i_2,\dots i_r} A^{(z)}(i_1, i_2, \dots i_r) \prod_{z_j=1} x_{i_j} \prod_{z_j=0} (1 - x_{i_j}).$$
Each $A^{(z)}$ is an array on V^r. Recall that $n = |V|$. We find the cut decomposition $B^{(z)}$ of each of the arrays $A^{(z)}$, as in Theorem 4. Each $B^{(z)}$ is the sum of at most $4/\epsilon^2$ cut arrays and we have (using (2))
$$||A^{(z)} - B^{(z)}||_C \leq \epsilon n^{r/2}||A^{(z)}||_F \leq \delta m/4. \qquad \forall z.$$

By the claim at the end of Section 2 it suffices to maximize the function $g(x)$ below to additive error $\delta m/2$:

$$g(x) = \sum_{z \in \{0,1\}^r} \sum_{i_1,i_2,\ldots i_r} B^{(z)}(i_1, i_2, \ldots i_r) \prod_{z_j=1} x_{i_j} \prod_{z_j=0} (1 - x_{i_j}). \tag{13}$$

Let

$$S_1, S_2, \ldots S_s$$

be all subsets of V defining the cut arrays we get in the decompositions of all the $A^{(z)}$. Note that $s \leq 4 \cdot 2^r/\epsilon^2$. Now suppose a particular cut array, say, $CUT(S_1, S_2, \ldots S_r; d)$ [2] occurs in the decomposition of a particular $A^{(z)}$, then we note that

$$\sum_{i_1,i_2,\ldots i_r} CUT(S_1, S_2, \ldots S_r; d) \prod_{z_j=1} x_{i_j} \prod_{z_j=0} (1 - x_{i_j})$$
$$= d \prod_{z_t=1} x(S_t) \prod_{z_t=0} (|S_t| - x(S_t)). \tag{14}$$

where here, as before, $x(S_t) = \sum_{i \in S_t} x_i$. Thus $g(x)$ is a function of just $\{x(S_t) : t = 1, 2, \ldots s\}$. Indeed, we have $g(x) = f(x(S_1), x(S_2), \ldots x(S_s))$, where f is a polynomial of degree at most r. We will need the fact that this f does not vary much as x changes (so that to evaluate the polynomials approximately, it will suffice to have the x approximately). To this end, (noting that both $x(S_t)$ and $|S_t| - x(S_t)$ are between 0 and n for all t), we get that when we replace one of the quantities $x(S_t)$ by a quantity $y(S_t)$, the value of the product changes by at most $n^{r-1}|x(S_t) - y(S_t)|$. Making these replacements one by one it follows that

$$\left| \prod_{z_t=1} x(S_t) \prod_{z_t=0} (|S_t| - x(S_t)) - \prod_{z_t=1} y(S_t) \prod_{z_t=0} (|S_t| - y(S_t)) \right| \leq rn^{r-1} \mathrm{Max}_{t=1}^{s} |x(S_t) - y(S_t)|.$$

By (8) of Theorem 4, we have that the sum of the absolute values of the coefficients of the cut arrays used in approximating one $A^{(z)}$ is at most $2||A^{(z)}||_F/(\epsilon n^{r/2})$. So, it follows that if x, y are two $n-$vectors satisfying

$$|x(S_t) - y(S_t)| \leq \nu n \qquad \text{for } t = 1, 2, \ldots s, \tag{15}$$

where $0 < \nu < 1$ will be specified later, we get (using (2)) that :

$$|g(x) - g(y)| \leq 2^r 2^{2^{r-1}} 2\sqrt{m} r \nu n^{r/2}/\epsilon \leq 2^{2^r} \sqrt{m} \nu n^{r/2}/\epsilon. \tag{16}$$

Choosing

$$\nu = \epsilon^2, \tag{17}$$

we get

$$(15) \quad \Longrightarrow \quad |g(x) - g(y)| \leq \delta m/4. \tag{18}$$

Thus it suffices to enumerate all possible $\{x(S_t) : t = 1, 2, \ldots s\}$, where each coordinate is within νn of the correct value. This can be done by Linear Programming in the required time, by associating each cell of the Venn diagram of the sets S_i with a variable, and by checking feasibility of each possible vector $\{x(S_t) : t = 1, 2, \ldots s\}$, where the value of each coordinate is specified up to νn by writing down the appropriate inequalities.

[2] The cut array need not involve the first r of the S_t 's; we just use $S_1, S_2, \ldots S_t$ for notational convenience.

5 Cut Norm of Random Sub-arrays

Let $A^{(z)}$ be the arrays on V^r defined in Section 2. We assume that the functions $f_1, f_2, \dots f_m$ comprising the input to the MAX-CSP problem are all distinct. Thus, as in (2), we get the first inequality below from which the second follows :

$$|A^{(z)}(i_1, i_2, \dots i_r)| \le 2^{2^r} \qquad ||A^{(z)}||_F^2 \le 2^{2^{r+1}} n^r. \tag{19}$$

From Theorem (4), we know that there is an approximation $B^{(z)}$ (which is the sum of a small number of cut arrays) to each $A^{(z)}$. We do not need these approximations in detail here. The main purpose of this section is to show that for a random subset J of V of cardinality $\Omega(\log(1/\epsilon)/\epsilon^4)$, the sub-array of $B^{(z)}$ induced by J, (namely $B^{(z)}$ restricted to J^r) is a good approximation to $A^{(z)}$ restricted to J^r. To simplify notation, we will let G stand for $A^{(z)} - B^{(z)}$. We will assume in this section that G satisfies the following conditions.

$$||G||_C \le \epsilon n^r \quad ||G||_\infty \le \frac{1}{\epsilon} 2^{2^{r+1}} \qquad ||G||_F \le 2^{2^r} n^{r/2}. \tag{20}$$

These are obtained from Theorem (4) with ϵ there replaced by $\epsilon/2^{2^r}$. These are the only properties of G we will use in this section. Here is the main theorem of this section.

Theorem 6. *Suppose G is an r-dimensional array on V^r satisfying (20). Let $\delta, \epsilon > 0$. Assume $n = |V| \ge \frac{10^8 r^{20}}{\delta^7 \epsilon^8} e^{10/\epsilon^2}$. Let J be a random subset of V of cardinality q, where,*

$$q \ge 10^6 r^{12} \frac{1}{\delta^5 \epsilon^4} \log\left(\frac{4}{\epsilon^2}\right).$$

Let H be the r-dimensional array obtained by restricting G to J^r. Then, we have with probability at least $1 - \delta$:

$$||H||_C \le 2^{2^{r+1}+9} \frac{\epsilon}{\sqrt{\delta}} q^r.$$

Before starting the formal proof, we give an intuitive description of it. In essence, what we want to prove is that if G has cut norm at most ϵn^r, (and also some bounds on its Frobenius and infinity norm), then, a random induced sub-array of G on q elements has cut norm at most $O(\epsilon q^r)$. Note that the reverse assertion that that if $||G||_C$ is high, then so is $||H||_C$ is much easier to prove - indeed, for this, we may just take the $S_1, S_2, \dots S_r \subseteq V$ achieving $||G||_C$ and argue just by the usual sampling theorems that $H(S_1 \cap J, S_2 \cap J, \dots S_r \cap J) \approx \frac{|J|^r}{n^r} ||G||_C$. Such a simple proof does not work for what we want here, since here we want to argue that the non-existence of a $S_1, S_2, \dots S_r$ achieving high $|G(S_1, S_2, \dots S_r)|$ implies the same for H. The general method of attack we use is to show that the number of candidate $S_1, S_2, \dots S_r$ we need to consider is not too high.

In more detail, the outline of the proof is as follows : Assume that J has already been picked. Suppose we pick in addition, r random subsets of J^{r-1} - $Q_1, Q_2, \dots Q_r$- each of cardinality $\Omega(1/\epsilon^2)$. Then, lemma 2 asserts that with high probability, there are subsets $Q_1' \subseteq Q_1$, $Q_2' \subseteq Q_2 \dots Q_r' \subseteq Q_r$ such that

$$H(\text{Pos}(Q_1'), \text{Pos}(Q_2'), \dots \text{Pos}(Q_r')) \approx H^+. \tag{21}$$

In other words, we need to consider only $2^{O(1/\epsilon^2)}$ candidate subsets of J to find the $S_1, S_2, \ldots S_r \subseteq J$ approximately maximizing $H(S_1, S_2, \ldots S_r)$ (not all $2^{O(|J|)}$ of them.) Now consider one fixed candidate - $Q_1', Q_2', \ldots Q_r'$. If now we could fix this candidate and assume that J was picked independently of this, (obviously we cannot), then we would have that $\mathrm{Pos}(Q_1') \cap J$ is a random subset of $\mathrm{Pos}(Q_1')$ (note that $\mathrm{Pos}(Q_1')$ is viewed as a subset of the whole V), $\mathrm{Pos}(Q_2') \cap J$ is a random subset of $\mathrm{Pos}(Q_2')$, $\ldots \mathrm{Pos}(Q_r') \cap J$ is a random subset of $\mathrm{Pos}(Q_r')$ and so by standard sampling theorems, we should have that with high probability the following holds :

$$G(\mathrm{Pos}(Q_1'), \mathrm{Pos}(Q_2'), \ldots \mathrm{Pos}(Q_r')) \approx \frac{|V|^r}{|J|^r} G(\mathrm{Pos}(Q_1') \cap J, \mathrm{Pos}(Q_2') \cap J, \ldots \mathrm{Pos}(Q_r') \cap J). \quad (22)$$

We will derive a quantitative version of this by applying the lemma 8 (to come) with G of that lemma defined from our G by zeroing out the entries outside $\mathrm{Pos}(Q_1') \times \mathrm{Pos}(Q_2') \times \ldots \mathrm{Pos}(Q_r')$.

Multiplying the failure probability in (22) with the number of possible subsets of the Q_i (which is $2^{O(1/\epsilon^2)}$), we also get that with high probability, (22) holds for every subset Q_1' of Q_1, Q_2' of Q_2 etc. If this holds rigorously, we would then clearly be able to infer from (21) and (22) that

$$G^+ \geq \frac{|V|^r}{|J|^r} H^+ - \text{ error }.$$

A similar inequality also will follow (along the same lines) for $(-G)^+$ and this would finish the proof.

The major problem is that J is not independent of $Q_1, Q_2, \ldots Q_r$; if it were (21) will not hold. ((21) needs $Q_1, Q_2, \ldots Q_r$ to be random subsets of J^{r-1}.) To tackle this, we adopt a method of proof reminiscent of the argument of Vapnik and Chervonenkis [18]. We consider a set J' which is J minus all the end points of $(r-1)$ tuples in $Q_1, Q_2, \ldots Q_r$. Noting that $|J| - |J'| \in O(1/\epsilon^2)$, we argue that we get roughly the same probability distributions if we pick, as we described already, J first and then $Q_1, Q_2, \ldots Q_r$ as random subsets of J^{r-1}, whence (21) holds as if we first pick J' and then $Q_1, Q_2, \ldots Q_r$ as random subsets of V^{r-1}, whence we have that (22) holds. Thus, we will see that we may actually use both (21) and (22) to get our result.

5.1 Two technical sampling lemmas

We start with two technical lemmas we need. The first lemma is a particular "large-deviations" result. While the proof is standard, it differs from the usual ones in its hypothesis which upper bound each real as well as the sum of squares. [We note that if we did not have the upper bound on the sum of squares, the upper bound one usually gets on the probability in the lemma depends on γ^2 rather than γ.]

Lemma 7. *Suppose $a_1, a_2, \ldots a_N$ are any reals with $|a_i| \leq M$ for all i and $\sum_{i=1}^N a_i^2 \leq \alpha$. Let $X_1, X_2, \ldots X_q$ be a sample of size q picked by sampling uniformly without replacement* *from the set* $\{a_1, a_2, \dots a_N\}$. *Then, for any real* $\gamma \geq \frac{2\alpha}{NM^2}$, *we have :*

$$\mathbf{Pr}\left(\left|\sum_{t=1}^{q} X_t - \frac{q}{N}\sum_i a_i\right| \geq \gamma M q\right) \leq 2e^{-\gamma q/4}.$$

Proof Let λ be a positive real to be chosen later. Let $\bar{a} = \frac{1}{N}\sum_{i=1}^{N} a_i$ and $b_i = a_i - \bar{a}$ and let $Y_1, Y_2, \dots Y_q$ be a sample of size q drawn with replacement from the same set of reals - $\{a_1, a_2, \dots a_N\}$. (To be used just in the proof.) Let $\Lambda = \gamma M q$.

$$\mathbf{Pr}\left(\sum_{t=1}^{q} X_t \geq q\bar{a} + \Lambda\right) \leq E\left(e^{\lambda \sum_t X_t}\right) e^{-\lambda q \bar{a}} e^{-\lambda\Lambda} \leq E\left(e^{\lambda \sum_t Y_t}\right) e^{-\lambda q \bar{a}} e^{-\lambda\Lambda}$$

the last inequality holds since e^x is a convex function - from Theorem 4 of [14]

$$= \left(E\left(e^{\lambda(Y_1 - \bar{a})}\right)\right)^q e^{-\lambda\Lambda}. = \frac{1}{N^q}\left(\sum_{i=1}^{N} e^{\lambda b_i}\right)^q e^{-\lambda\Lambda}.$$

The b_i satisfy the constraints $\sum_i b_i^2 \leq \sum_i a_i^2 \leq \alpha$ and $|b_i| \leq 2M$. The maximum of the last expression subject to these two constraints is attained when N_0 =Min($N, \frac{\alpha}{4M^2}$) of the b_i 's are $2M$ each and the rest are zero. Thus, we have, by choosing $\lambda = 1/(2M)$ in the above,

$$\mathbf{Pr}\left(\sum_{t=1}^{q} X_t \geq q\bar{a} + \Lambda\right) \leq \frac{1}{N^q}[N_0 e + N - N_0]^q e^{-\Lambda/(2M)} \leq (1 + \frac{\alpha}{2NM^2})^q e^{-\Lambda/(2M)} \leq e^{-\gamma q/4},$$

using $(1 + (\alpha/4NM^2)) \leq e^{\alpha/4NM^2}$. This bounds the probability of $\sum X_t$ being too large. To bound the probability of this sum being too negative, we just use the same argument with the set of a_i replaced by the set of $-a_i$. This then yields the lemma. □

The next lemma says that a "induced" sub-array estimates the sum of all elements of a large array well.

Lemma 8. *Let t be a positive integer multiple of r satisfying $t \geq 2^{r+8}\log(1/\epsilon)/\epsilon^2$. Let I be a random subset of V of cardinality t. With probability at least $1 - 8e^{-t\epsilon^2/16r}$ the following holds :*

$$\left|G(V^r) - \frac{n^r}{t^r}G(I^r)\right| \leq \epsilon n^r 2^{2^{r+1}+4}.$$

Proof Let

$$\gamma = \epsilon^2/2 \qquad M = \frac{1}{\epsilon}2^{2^{r+1}}.$$

Note that $||G||_\infty \leq M$. Let X denote a set of t/r elements of V^r picked in i.i.d. trials, each uniformly. (X is an auxiliary set which is only used for the proof.) With probability at least $1 - \frac{10t^2}{n}$, the set $end(X)$ of end points of elements of X is of cardinality t; we will henceforth assume this happens after paying the failure probability. Let

$$Bad = \left\{X : \left|\sum_{w \in X} G(w) - \frac{t}{rn^r}G(V^r)\right| \geq \gamma M t/r\right\}.$$

From Lemma 7, (with α there equal to $2^{2^{r+1}}n^r$) we get that

$$|Bad| \leq 2e^{-\gamma t/4r}\binom{n^r}{t/r}.$$

For an $I \subseteq V$ with $|I| = t$, let $f(I)$ denote the set of X with $end(X) = I$. Let I_1 be the set of $w \in I^r$ with r distinct end points. Since each $w \in I_1$ belongs to precisely $(t-r)!/((t/r)-1)!$ X 's in $f(I)$, we have that

$$\sum_{X \in f(I)} \sum_{w \in X} G(w) = \frac{(t-r)!}{(\frac{t}{r}-1)!} \sum_{w \in I_1} G(w) = \frac{(t-r)!}{(\frac{t}{r}-1)!}(G(I^r) + \Delta), \text{ where, } |\Delta| \in Mr^2t^{r-1}. \tag{23}$$

Each X with $|end(X)| = t$ clearly belongs to $f(I)$ for precisely one I. Noting that $\binom{n^r}{t/r}(t/r)! \leq 2t!\binom{n}{t}$, we have that the event defined below has the claimed probability bound :

$$E_0(I) : |Bad \cap f(I)| \leq 2e^{-\gamma t/8r}\frac{t!}{(t/r)!} \text{ satisfies}$$
$$\mathbf{Pr}(E_0(I)) \geq 1 - 2e^{-\gamma t/8r}$$

Now, we have

$$\left|\sum_{w \in X} G(w) - \frac{t}{rn^r}G(V^r)\right| \leq \gamma\frac{Mt}{r} \qquad \text{for } X \notin Bad$$
$$\left|\sum_{w \in X} G(w) - \frac{t}{rn^r}G(V^r)\right| \leq \frac{2Mt}{r} \qquad \text{for } X \in Bad.$$

$$\begin{aligned} \text{So, } \left|\sum_{X \in f(I)} \sum_{w \in X} G(w) - \frac{t}{rn^r}G(V^r)\frac{t!}{(t/r)!}\right| &\leq \frac{\gamma Mt}{r}\frac{t!}{(t/r)!} + 2|Bad \cap f(I)|\frac{Mt}{r} \\ \text{Under } E_0(I) \qquad &\leq \frac{t!}{(t/r)!}\frac{Mt}{r}(\gamma + 2e^{-\gamma t/8r}) \leq \frac{t!}{(t/r)!}\frac{Mt}{r}1.1\gamma. \end{aligned}$$

(The last inequality also uses the lower bound on t in the hypothesis of the Lemma.) Thus, using (23), we get :

$$E_0(I) \Longrightarrow G(I^r) = \frac{t!}{(t-r)!}\frac{1}{n^r}G(V^r) - \Delta + \Delta', \text{ where, } |\Delta'| \leq 1.1t^r\gamma M.$$

So, we have

$$\left|\frac{n^r}{t^r}G(I^r) - G(V^r)\right| \leq \left|\frac{t^r t!}{(t-r)!} - 1\right| |G(V^r)| + |\Delta| + |\Delta'|.$$

From this, the lemma follows. $\square$

5.2 Proof of Theorem 6

First we have that $E(||H||_F^2) = \frac{q^r}{n^r}||G||_F^2$, so using Markov inequality, we have that event

$$E_1 : ||H||_F \leq \frac{2}{\sqrt{\delta}} \frac{q^{r/2}}{n^{r/2}}||G||_F \text{ has } \mathbf{Pr}(E_1) \geq 1 - (\delta/4). \tag{24}$$

Let $p = 100r^4/(\delta^2\epsilon^2)$. Let $Q_1, Q_2, \ldots Q_r$ be r randomly picked subsets of J^{r-1}, (independently, each uniformly picked), each of cardinality p. We apply Lemma 2 to H (not to G. So, now $N = q^r$). So, with probability at least $1 - (\delta/5)$ (using (24))

$$\begin{aligned} &\exists Q_1' \subseteq Q_1, \exists Q_2' \subseteq Q_2, \ldots \exists Q_r' \subseteq Q_r, \\ &G(\text{Pos}(Q_1') \cap J, \text{Pos}(Q_2') \cap J, \ldots \text{Pos}(Q_r') \cap J) \\ &\geq H^+ - \frac{\epsilon}{2} q^{r/2}||H||_F \geq H^+ - \frac{\epsilon}{\sqrt{\delta}} q^r ||G||_F/n^{r/2}. \end{aligned} \tag{25}$$

Here, we mean by $\text{Pos}(Q_1')$ the set $\{z \in V : G(z, Q_1') > 0\}$; so, $\text{Pos}(Q_1')$ is a subset of V, not just J. Let J' be obtained from J by removing the at most $r(r-1)p$ end points of the elements of $Q_1 \cup Q_2 \cup \ldots Q_r$.

We will make crucial use of the fact that the following two different methods of picking $J, Q_1, Q_2, \ldots Q_r$ produce nearly the same joint probability distribution on them :

(i) As above, pick J to be a random subset of V of cardinality q and then pick $Q_1, Q_2, \ldots Q_r$ to be independent random subsets of J^{r-1} each of cardinality p. Let $P^{(i)}(J, Q_1, Q_2, \ldots Q_r)$ be the probability that we pick $J, Q_1, Q_2, \ldots Q_r$ in this experiment. Then, clearly, for each $J, Q_1, Q_2, \ldots Q_r$ with $|J| = q, Q_1, Q_2, \ldots Q_r \subseteq J^{r-1}, |Q_i| = p$, we have

$$P^{(i)}(J, Q_1, Q_2, \ldots Q_r) = \left(\binom{n}{q} \binom{q^{r-1}}{p}^r \right)^{-1}.$$

(ii) Pick independently (of each other) r random subsets $\tilde{Q}_1, \ldots \tilde{Q}_r$ of V^{r-1} of cardinality p each. Then, pick J' to be a random subset of V of cardinality $q - r(r-1)p$ (independently of $\tilde{Q}_i$'s). Let $\tilde{J} = J' \cup$ (the set of all end points of elements of $\tilde{Q}_1 \cup \tilde{Q}_2 \ldots \tilde{Q}_r$). Let $P^{(ii)}(\tilde{J}, \tilde{Q}_1, \ldots \tilde{Q}_r)$ be the probabilities here.

Define E_2 to be the event that all $pr(r-1)$ end points of the elements in $Q_1, Q_2, \ldots Q_r$ are distinct and let E_3 be the event that all the end points of $\tilde{Q}_1, \tilde{Q}_2, \ldots \tilde{Q}_r$ are distinct and none of them is in J'. It is easy to see by direct calculation that conditioned on the events E_2, E_3, $P^{(i)}$ and $P^{(ii)}$ are exactly equal. We wish to show that $P^{(i)}(E_2)$ is close to 1. To this end, in (i), having picked J, we pick $pr(r-1)$ independent identically distributed samples, each uniformly from J. The probability that some pair of them is equal is at most

$$\binom{pr(r-1)}{2} \frac{1}{q} \leq \frac{p^2 r^4}{q} \leq \frac{\delta}{8}.$$

Thus, $1 - P^{(i)}(E_2) \leq \delta/8$. Also,

$$1 - P^{(ii)}(E_3) \leq \binom{pr(r-1)}{2} \frac{1}{n} + pr(r-1)\frac{q}{n} \leq \delta/4.$$

So we have that the following inequality which we will use shortly :

$$||P^{(i)} - P^{(ii)}||_{\mathrm{TV}} \leq 3\delta/8, \tag{26}$$

(where $||P^{(i)} - P^{(ii)}||_{\mathrm{TV}}$ denotes the usual "total variation" distance between the two probability distributions, namely, the maximum over all subsets X of the sample space of the quantity $|P^{(i)}(X) - P^{(ii)}(X)|$.)

Pick $\tilde{Q}_1, \tilde{Q}_2, \ldots \tilde{Q}_r$ as in $P^{(ii)}$. For now, fix a particular collection of subsets $Q'_1 \subseteq Q_1, Q'_2 \subseteq Q_2, \ldots Q'_r \subseteq Q_r$. Define an array G' by :

$$\begin{aligned} G'(i_1, i_2, \ldots i_r) &= G(i_1, i_2, \ldots i_r) \quad (i_1, i_2, \ldots i_r) \in \mathrm{Pos}(Q'_1) \times \mathrm{Pos}(Q'_2) \times \ldots \mathrm{Pos}(Q'_r) \\ G'(i_1, i_2, \ldots i_r) &= 0 \text{ otherwise .} \end{aligned}$$

Note that $||G'||_F \leq ||G||_F$. Now, we pick $J' \subseteq V$ of cardinality $q - r(r-1)p$ independently of $\tilde{Q}_1, \tilde{Q}_2, \ldots \tilde{Q}_r$ as in $P^{(ii)}$. Applying the Lemma 8 to G' (not to G), with t of that lemma set to $q - r(r-1)p$ and I of that lemma set to J', we get the claimed bounds for the probabilities of the events defined below :

$$\text{Let } E_8(J', Q'_1, Q'_2, \ldots Q'_r) :$$

$$\left| G(\mathrm{Pos}(Q'_1), \mathrm{Pos}(Q'_2), \ldots \mathrm{Pos}(Q'_r)) - \frac{n^r}{(q - r(r-1)p)^r} G(\mathrm{Pos}(Q'_1) \cap J', \mathrm{Pos}(Q'_2) \cap J', \ldots, \mathrm{Pos}(Q'_r) \cap J') \right|$$

$$\leq \epsilon n^r 2^{2^{r+1}+4}$$

$$\text{Then, } P^{(ii)}(E_8(J', Q'_1, Q'_2, \ldots Q'_r)) \geq 1 - 8e^{-q\epsilon^2/32r} \geq 1 - \frac{\delta}{8} e^{-5pr}.$$

Now using the fact that for one choice of $\tilde{Q}_1, \tilde{Q}_2, \ldots \tilde{Q}_r$, there are 2^{pr} choices of $Q'_1, Q'_2, \ldots Q'_r$, we get :

$$E_9(J', \tilde{Q}_1, \tilde{Q}_2, \ldots \tilde{Q}_r) \;:\; \forall Q'_1 \subseteq \tilde{Q}_1, \forall Q'_2 \subseteq \tilde{Q}_2, \ldots \forall Q'_r \subseteq \tilde{Q}_r, \quad E_8(J', Q'_1, Q'_2, \ldots Q'_r)$$
$$P^{(ii)}(E_9(J', Q_1, Q_2, \ldots Q_r)) \geq 1 - \tfrac{\delta}{8}.$$

Now, let J be the union of J' and the end points of elements of $\tilde{Q}_i$'s. Noting that $q^r \leq (1 + \epsilon^2)(q - r(r-1)p)^r$ and

$$|G(\mathrm{Pos}(Q'_1) \cap J', \mathrm{Pos}(Q'_2) \cap J', \ldots, \mathrm{Pos}(Q'_r) \cap J') - G(\mathrm{Pos}(Q'_1) \cap J, \mathrm{Pos}(Q'_2) \cap J, \ldots, \mathrm{Pos}(Q'_r) \cap J)|$$

$$\leq \epsilon^2 q^r ||G||_\infty,$$

we get (using also (26)) :

$$\text{Let } E_{10}(J, Q_1, Q_2, \ldots Q_r) : \forall Q'_1 \subseteq Q_1, \forall Q'_2 \subseteq Q_2, \ldots \forall Q'_r \subseteq Q_r$$
$$\left| G(\mathrm{Pos}(Q'_1), \mathrm{Pos}(Q'_2), \ldots \mathrm{Pos}(Q'_r)) - \tfrac{n^r}{(q - r(r-1)p)^r} G(\mathrm{Pos}(Q'_1) \cap J, \mathrm{Pos}(Q'_2) \cap J, \ldots, \mathrm{Pos}(Q'_r) \cap J) \right|$$
$$\leq \epsilon n^r 2^{2^{r+1}+5}$$
$$P^{(i)}(E_{10}(J, Q_1, Q_2, \ldots Q_r)) \geq 1 - \tfrac{\delta}{2}. \tag{27}$$

Under $E_{10}(J, Q_1, Q_2, \dots Q_r)$, we have from (25) that

$$\exists Q_1' \subseteq Q_1, \exists Q_2' \subseteq Q_2 \dots G(\text{Pos}(Q_1'), \text{Pos}(Q_2'), \dots \text{Pos}(Q_r'))$$
$$\geq \frac{n^r}{(q - r(r-1))^r} H^+ - \frac{2\epsilon}{\sqrt{\delta}} n^{r/2} ||G||_F - \epsilon n^r 2^{2^{r+1}+5}$$
$$\geq \frac{n^r}{q^r} H^+ - \frac{\epsilon}{\sqrt{\delta}} n^r 2^{2^{r+1}+8}.$$

Thus, we get that with probability at least $1 - \frac{\delta}{2}$, the following holds:

$$G^+ \geq \frac{n^r}{q^r} H^+ - \frac{\epsilon}{\sqrt{\delta}} n^r 2^{2^{r+1}+8}.$$

By an exactly identical argument applied to $-G$, we get also that with probability at least $1 - \delta/2$,

$$(-G)^+ \geq \frac{n^r}{q^r} (-H)^+ - \frac{\epsilon}{\sqrt{\delta}} n^r 2^{2^{r+1}+8}.$$

From the last two statements, the Theorem follows.

$\square$

6 Random sub-programs of Linear Programs

In this section, we prove a result about Linear Programs which we will use later. The result may be of independent interest. It says that for a Linear Program on n variables, each constrained to be between 0 and 1, we can make some assertion about the optimal value based on the optimal value of a small sub-program obtained by picking at random a small number of variables. We first state a simple theorem which illustrates the essential proof technique. Then we prove a more complicated (technical) theorem which is the one we will use.

We remark that having the variables bounded between 0 and 1 is crucial; if the "scales" of the variables were different, it is intuitively clear that uniform random sampling will not yield a good approximation.

Theorem 9. *Suppose* [3]

$$\alpha > Max \sum_{j=1}^{n} c_j x_j$$
$$\sum_{j=1}^{n} U_j x_j \leq v \qquad ; \qquad 0 \leq x_j \leq 1,$$

[3]We write the line below as shorthand for " the optimal value of the linear program is less than α". If the linear program is infeasible, we let the optimal value be $-\infty$.

where each U_j is an $m-$ vector. Suppose q is a positive integer and Q is a random subset of $\{1,2,\dots n\}$ of cardinality q. Then, for any positive real number λ, with probability at least $1-4e^{-\lambda^2/4}$, we have :

$$\frac{q}{n}\alpha+\lambda\sqrt{q}||c||_\infty \geq Max\sum_{j\in Q} c_j x_j$$

$$\sum_{j\in Q} U_j x_j \leq \frac{q}{n}v-\lambda\sqrt{q}||U||_\infty \qquad ; \qquad 0\leq x_j\leq 1, j\in Q.$$

Remark Before we start the proof of the Theorem, we give the reader an intuitive idea of the reasoning. First note that a "reverse" of the Theorem which asserts that if the whole LP has a high optimal value, then the induced LP on Q has a high optimal value is much easier to prove - we could just take the optimal solution to the whole LP and argue just by random sampling that the induced solution on Q provides a reasonable solution to the LP induced on Q. Here however, we want to show that the non-existence of a good solution to the whole LP implies the same for the random induced LP on Q. Luckily, this is also not too hard for LP's, because LP duality says that the non-existence of a good solution to the whole LP is equivalent to the existence of a certain solution to the dual LP. We can then take this solution and it induces a solution to the corresponding induced problem on Q.

Proof By Linear Programming Duality, there exist a non-negative $m-$vector u and a non-negative real number β such that

$$\nexists x : \sum_{j=1}^{n}(uU_j-\beta c_j)x_j \leq uv-\beta\alpha \quad ; \quad 0\leq x_j\leq 1$$

$$\Longrightarrow \sum_{j=1}^{n}(uU-\beta c)_j^- > uv-\beta\alpha,$$

the last since one linear inequality has a solution over $x, 0\leq x_j\leq 1$ iff setting to 1 the variables with a negative coefficient in the inequality and setting the rest to 0 satisfies it. Noting that

$$|(uU-\beta c)_j| \leq \sum_i u_i||U||_\infty + \beta||c||_\infty \forall j,$$

we get that the event below has the claimed probability :

$$E_{11}: \sum_{j\in Q}(uU-\beta c)_j^- > \frac{q}{n}(uv-\beta\alpha)-\lambda\sqrt{q}\left((\sum_i u_i)||U||_\infty+\beta||c||_\infty\right)$$

$$\mathbf{Pr}(E_{11})\geq 1-4e^{-\lambda^2/4}.$$

Now, it is easy to see that event E_{11} implies the conclusion of the theorem. [If not, taking the solution x attaining the optimal value in the linear program and adding up the inequalities after multiplying by the same u,β produces a contradiction to E_{11}.] This completes the proof of the theorem. The next theorem is stronger in the case when $\sum_j c_j^2 \ll n||c||_\infty^2$, i.e., when a few of the $|c_j|$ are much larger than the average.

Theorem 10. *Suppose*

$$\alpha > Max \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} U_j x_j \leq v \qquad ; \qquad 0 \leq x_j \leq 1,$$

as before and, in addition,

$$\sum_{j=1}^{m} c_j^2 \leq \alpha_2 \qquad ||c||_\infty \leq M_2.$$

Suppose q is a positive integer and Q is a random subset of $\{1, 2, \ldots n\}$ of cardinality q. Then, for any positive real number

$$\gamma \in \left[\frac{4\alpha_2}{nM_2^2} \ , \ 100\right],$$

we have that with probability at least $1 - 4e^{-\gamma q/4}$:

$$\frac{q}{n}\alpha + 2\gamma q M_2 > Max \sum_{j \in Q} c_j x_j$$

$$\sum_{j \in Q} U_j x_j \leq \frac{q}{n} v - 2\sqrt{\gamma} q ||U||_\infty \qquad ; \qquad 0 \leq x_j \leq 1, j \in Q.$$

Proof Arguing as in the last theorem, we again get that $\sum_{j=1}^{n} (uU - \beta c)_j^- > uv - \beta\alpha$. Let $V' = \{j : (uU - \beta c)_j < 0\}$. The random variable

$$\begin{aligned} X &= \sum_{j \in Q} (uU - \beta c)_j^- = \sum_{j \in Q} (uU - \beta c)_j \chi(j \in V') \\ &= \sum_{j \in Q} (uU)_j \chi(j \in V') + \sum_{j \in Q} (-\beta c_j) \chi(j \in V') \\ &= X_1 + X_2 \text{ say, respectively .} \end{aligned}$$

Now X_1 can be written as the sum of q independent random variables, each at most $(\sum_i u_i)||U||_\infty$ in absolute value. So, we have by standard Hoeffding inequality,

$$E_{12} : \left| X_1 - \frac{q}{n} \sum_{j \in V'} (uU)_j \right| \leq \sqrt{\gamma} q (\sum_i u_i) ||U||_\infty \text{ has}$$
$$\mathbf{Pr}(E_{12}) \geq 1 - 2e^{-\gamma q}.$$

To X_2, we will apply our sampling Lemma 7 with M of that lemma equal to M_2 and α of that lemma equal to α_2 to get that

$$E_{13} : \left| X_2 - \frac{q}{n} \beta \sum_{j \in V'} c_j \right| \leq 2\beta\gamma q M_2 \text{ has}$$
$$\mathbf{Pr}(E_{13}) \geq 1 - 2e^{-\gamma q}.$$

It is easy to see that under E_{12} and E_{13}, we have the conclusion of the theorem.

$\square$

7 Proof of Main Theorem (Theorem 1)

Recall that we are given a set F of m distinct functions $f_1, f_2, \ldots f_m$, each of r variables among $\{x_1, x_2, \ldots x_n\}$. We are also given $\epsilon > 0$ and we wish to find the maximum number of satisfiable functions to additive error ϵn^r. In this section, we will use $O(\cdot)$ to hide constant factors independent of ϵ (and of course n). These factors will depend only upon r. [It would obscure the main points if we put in all the constants, so we have not done that.] We will actually only achieve an error of $O(\epsilon n^r)$. [By suitably adjusting ϵ, we can then achieve an error of actually ϵn^r.]

As in Section 2, define $A^{(z)}, A$ and $P(x)$. Note that since the $f_1, f_2, \ldots f_m$ are distinct, we have (see (2)),

$$|A^{(z)}(i_1, i_2, \ldots i_r)| \in O(1) \qquad ||A||_F^2 \in O(n^r). \tag{28}$$

Now we use Theorem 4 to assert that for each z, there exist $B^{(z)}$, which is the sum of $s \in O(1/\epsilon^2)$ cut arrays and which satisfies

$$||B^{(z)} - A^{(z)}||_C \leq \epsilon n^r. \tag{29}$$

We do not find the $B^{(z)}$ here; all we need in this section is the fact that they exist. Recall the definition of $P(x)$ from (1); for convenience, we change the definition slightly here and normalize by dividing by n^r. The new definition (which will cause no confusion) of $P(x)$ and a similar polynomial g which we use here are as follows :

$$P(x) = \frac{1}{n^r} \sum_{z \in \{0,1\}^r} \sum_{i_1, i_2, \ldots i_r} A^{(z)}(i_1, i_2, \ldots i_r) \prod_{j: z_j = 1} x_{i_j} \prod_{j: z_j = 0} (1 - x_{i_j}).$$

$$g(x) = \frac{1}{n^r} \sum_{z \in \{0,1\}^r} \sum_{i_1, i_2, \ldots i_r} B^{(z)}(i_1, i_2, \ldots i_r) \prod_{j: z_j = 1} x_{i_j} \prod_{j: z_j = 0} (1 - x_{i_j}).$$

$$(29) \implies \max_{x \in \{0,1\}^n} |P(x) - g(x)| \in O(\epsilon). \tag{30}$$

Suppose that the sets involved in defining all the cut arrays in the approximations of all $B^{(z)}$ are $S_1, S_2, \ldots S_s$. (We still have $s \in O(1/\epsilon^2)$.)

Let now Q be a random subset of V of cardinality q as in the statement of the Theorem.

We will denote by $\tilde{A}^{(z)}$ the sub-array of $A^{(z)}$ on Q^r. Similarly for $B^{(z)}$. From, Theorem 6, (with $q = c\frac{1}{\epsilon^4}\log(1/\epsilon)$ for a high enough constant c), we see that the following event has the claimed probability :

$$E_{16} : ||\tilde{A}^{(z)} - \tilde{B}^{(z)}||_C \in O(\epsilon q^r) \qquad \text{satisfies} \qquad \mathbf{Pr}(E_{16}) \geq \frac{99}{100}. \tag{31}$$

Define

$$\tilde{P}(x) = \frac{1}{q^r} \sum_{z\in\{0,1\}^r} \sum_{i_1,i_2,\dots i_r\in Q} A^{(z)}(i_1,i_2,\dots i_r) \prod_{j:z_j=1} x_{i_j} \prod_{j:z_j=0} (1-x_{i_j}).$$

$$\tilde{g}(x) = \frac{1}{q^r} \sum_{z\in\{0,1\}^r} \sum_{i_1,i_2,\dots i_r\in Q} B^{(z)}(i_1,i_2,\dots i_r) \prod_{j:z_j=1} x_{i_j} \prod_{j:z_j=0} (1-x_{i_j}). \tag{32}$$

$$E_{16} \Longrightarrow \max_{\{x_j\in\{0,1\},j\in Q\}} |\tilde{P}(x) - \tilde{g}(x)| \in O(\epsilon). \tag{33}$$

Recall that $\mathrm{Max}(F)$ denotes the maximum number of functions among F which can be simultaneously set to 1. Also recall that F^Q denotes the subset of the functions involving only the variables in Q. We have

$$\frac{1}{n^r}\mathrm{Max}(F) = \mathrm{Max}_{x\in\{0,1\}^n} P(x) \qquad \frac{1}{q^r}\mathrm{Max}(F^Q) = \mathrm{Max}_{x_j\in\{0,1\},j\in Q}\tilde{P}(x).$$

So from (30) and (33), to prove the Theorem, it suffices to show that

$$\left|\mathrm{Max}_{x_j\in\{0,1\},j\in V} \quad g(x) - \mathrm{Max}_{x_j\in\{0.1\},j\in Q} \quad \tilde{g}(x)\right| \in O(\epsilon). \tag{34}$$

To prove this, we will exploit the special structure of $g, \tilde{g}$.

We first need a simple technical fact :

Claim 1

$$E_{15} : \left|\frac{1}{n}|S_t| - \frac{1}{q}|S_t\cap Q|\right| \le \epsilon^2 \textit{ for } \quad t = 1,2,\dots s \quad \mathbf{Pr}(E_{15}) \ge 1 - 4se^{-\epsilon^4 q/4}.$$

Proof The random variable $|\frac{1}{n}|S_t| - \frac{1}{q}|S_t\cap Q||$ has expectation 0 and changes by at most $1/q$ when only one of the q random choices to select Q (each choice picks one element of Q) is changed. So, the claim follows by standard Martingale inequality.

□

Arguing as in Section 4, we see that $g(x)$ is the sum of $O(1/\epsilon^2)$ terms, each of the form $g_1(x)$ below and similarly, $\tilde{g}(x)$ is the sum of **corresponding** terms - $\tilde{g}_1(x)$:

$$g_1(x) = d \prod_{t:z_t=1} \frac{1}{n} x(S_t) \prod_{t:z_t=0} \frac{1}{n}(|S_t| - x(S_t))$$

$$\tilde{g}_1(x) = d \prod_{t:z_t=1} \frac{1}{q} x(S_t\cap Q) \prod_{t:z_t=0} \frac{1}{q}(|S_t\cap Q| - x(S_t\cap Q)).$$

($g_1(x)$ does not have to involve the first r S_t 's. It is only for notational convenience that we have used this here.) Thus, $x(S_1), x(S_2), \dots x(S_s)$ determine $g(x)$ and similarly $x(S_1\cap Q), x(S_2\cap Q), \dots x(S_s\cap Q)$ determine $\tilde{g}(x)$.

Denote by $h(x)$ the $s-$ vector $(\frac{1}{n}x(S_1), \frac{1}{n}x(S_2), \dots \frac{1}{n}x(S_s))$ (for an $n-$ vector x) and similarly by $\tilde{h}(x)$ the $s-$ vector $(\frac{1}{q}x(S_1\cap Q), \frac{1}{q}x(S_2\cap Q), \dots \frac{1}{q}x(S_s\cap Q))$ (for a $q-$ vector x with components for each $j \in Q$). We will approximate $g(x)$ by a piece-wise linear function, where, each piece will comprise of all the x 's for which the $h(x)$ are close. More precisely, we will use a parameter η - which will be $\Theta(\epsilon)$. Let $\mathcal{A}$ be the set of integer multiples of η in the range $(0,1)$. For each $b \in \mathcal{A}^s$, define

$$I(b,\eta) = \{x : |h(x) - b|_\infty \le 2\eta\} \qquad \tilde{I}(b,\eta) = \{x : |\tilde{h}(\tilde{x}) - b|_\infty \le \eta\}.$$

[Note that $I(b,\eta)$ is defined with 2η, whereas $\tilde{I}(b,\eta)$ is defined with only η, a difference we will make use of later.] The lemma below asserts the existence of the piece-wise linear approximation; we do not need to find it. Note that the "same" approximation works on V as well as on Q. Its proof will take up most of this section.

Lemma 11. *For a suitable choice of $\eta \in \Theta(\epsilon)$, for each fixed $b \in \mathcal{A}^s$, there exist two linear functions $l(x) = l_0 + \sum_{j=1}^n l_j x_j$ and $\tilde{l}(x) = \tilde{l}_0 + \sum_{j \in Q} \tilde{l}_j x_j$ such that*

$$|g(x) - l(x)| \in O(\epsilon)\ \forall x \in I(b,\eta) \qquad |\tilde{g}(x) - \tilde{l}(x)| \in O(\epsilon),\ \forall x \in \tilde{I}(b,\eta).$$

$$E_{16} \textit{ and } E_{15} \implies \left|\tilde{l}_0 + \sum_{j\in Q} \tilde{l}_j x_j - l_0 - \frac{n}{q}\sum_{j \in Q} l_j x_j\right| \in O(\epsilon) \forall x \in \tilde{I}(b,\eta).$$

Also, $|l_j| \in O(1/n\epsilon) \quad \forall j$ and $\sum_{j=1}^n l_j^2 \in O(1/n)$.

Proof

On each $I(b,\eta)$, $b \in \mathcal{A}$, we will approximate $g(x)$ by a linear function by approximating each term $g_1(x)$ by a linear function $g_2(x)$ and then adding up over all terms. To this end, we may write, (with $\mu(z) = (-1)^{|\{j:z_j=1\}|}$):

$$\begin{aligned} g_1(x) &= d \prod_{t:z_t=1} \frac{1}{n} x(S_t) \prod_{t:z_t=0} \frac{1}{n}(|S_t| - x(S_t)) \\ &= \mu(z) d \prod_{t=1}^r \left(\frac{1}{n}|S_t|(1-z_t) - b_t + (b_t - \frac{1}{n}x(S_t))\right) = g_2(x) + \Delta(x), \end{aligned}$$

where, expanding the above product,

$$g_2(x) = \mu(z) d \prod_{t=1}^r (\frac{1}{n}|S_t|(1-z_t) - b_t) + d\mu(z) \sum_{t=1}^r (b_t - \frac{1}{n}x(S_t)) \prod_{t' \neq t} (\frac{1}{n}|S_{t'}|(1-z_{t'}) - b_{t'})$$

$$|\Delta(x)| \le 4|d|\eta^2 2^r \ \forall x \in I(b,\eta),$$

the last because $\Delta(x)$ is the sum of $2^r - r - 1$ terms, namely the quadratic and higher degree terms in the expansion of the above expression; each term is the product of at least 2 factors of the form $b_t - \frac{1}{n}x(S_t)$, which is at most 2η in absolute value and other terms are of the form $\frac{1}{n}|S_t|(1-z_t) - b_t$ which is at most 1 in absolute value. We may rewrite the *linear function* $g_2(x)$ as :

$$g_2(x) = \mu(z) d (c_0 + \sum_{t=1}^r c_t x(S_t)) \qquad \text{where,}$$

$$c_0 = \prod_{t=1}^{r}(\frac{1}{n}|S_t|(1-z_t)-b_t) + \sum_{t=1}^{r} b_t \prod_{t'\neq t}(\frac{1}{n}|S_{t'}|(1-z_{t'})-b_{t'})$$
$$c_t = -\frac{1}{n}\prod_{t'\neq t}\left(\frac{1}{n}|S_{t'}|(1-z_{t'})-b_{t'}\right).$$

Proceeding exactly similarly, we get that

$$\tilde{g}_1(x) = d\mu(z)\prod_{t=1}^{r}\left(\frac{1}{q}|S_t\cap Q|(1-z_t)-b_t+(b_t-\frac{1}{q}x(S_t\cap Q))\right) = \tilde{g}_2(x)+\tilde{\Delta}(x),$$

where

$$\tilde{g}_2(x) =$$
$$\mu(z)d\prod_{t=1}^{r}(\frac{1}{q}|S_t\cap Q|(1-z_t)-b_t) + d\mu(z)\sum_{t=1}^{r}(b_t-\frac{1}{q}x(S_t\cap Q))\prod_{t'\neq t}(\frac{1}{q}|S_{t'}\cap Q|(1-z_{t'})-b_{t'})$$
$$|\tilde{\Delta}(x)| \le |d|\eta^2 2^r \ \forall x\in \tilde{I}(b,\eta).$$

We may again rewrite $\tilde{g}_2(x)$ as :

$$\tilde{g}_2(x) = \mu(z)d(\tilde{c}_0+\sum_{t=1}^{r}\tilde{c}_t x(S_t\cap Q)) \qquad \text{where,}$$
$$\tilde{c}_0 = \prod_{t=1}^{r}(\frac{1}{q}|S_t\cap Q|(1-z_t)-b_t) + \sum_{t=1}^{r} b_t\prod_{t'\neq t}(\frac{1}{q}|S_{t'}\cap Q|(1-z_{t'})-b_{t'})$$
$$\tilde{c}_t = -\frac{1}{q}\prod_{t'\neq t}\left(\frac{1}{q}|S_{t'}\cap Q|(1-z_{t'})-b_{t'}\right) \text{ for } t=1,2,\ldots r.$$

We will now prove some bounds between $c_t, \tilde{c}_t$.

Lemma 12. *Under E_{15}, we have :*

$$|c_0-\tilde{c}_0| \in O(\epsilon^2) \qquad |\tilde{c}_t - \frac{n}{q}c_t| \in O(\epsilon^2/q) \qquad \textit{for } t=1,2,\ldots r.$$

Proof For $t=1,2,\ldots r$, note that under E_{15},

$$\left|\frac{1}{n}|S_{t'}|(1-z_{t'})-b_{t'}-\frac{1}{q}|S_{t'}\cap Q|(1-z_{t'})+b_{t'}\right| \le \epsilon^2.$$

Also, each of $|\frac{1}{n}|S_{t'}|(1-z_{t'})-b_{t'}|$, $|\frac{1}{q}|S_{t'}\cap Q|(1-z_{t'})-b_{t'}|$ is at most 1. Now, we can write nc_t as the product of $r-1$ reals - call them $a_1,a_2,\ldots a_{r-1}$, each of absolute value at most 1; and similarly, $q\tilde{c}_t$ is the product of $r-1$ reals - call them $b_1,b_2,\ldots b_{r-1}$, each of absolute value at most 1 and we have $|a_i-b_i|\le\epsilon^2$. Consider for the moment, the function $f(\lambda)=\prod_{t=1}^{r}(\lambda a_t+(1-\lambda)b_t)$ and write $f(0)-f(1)$ as $\int_0^1 \frac{df}{d\lambda}d\lambda$; from this it follows that

$$|nc_t - q\tilde{c}_t| = |f(1)-f(0)| \in O(\epsilon^2).$$

The difference between c_0 and $\tilde{c}_0$ is bounded similarly.

$\square$

Lemma 12 implies that under E_{15}, we have

$$\forall x \in \tilde{I}(b,\eta), \left| d(c_0 + \frac{n}{q}\sum_{t=1}^{r} c_t x(S_t \cap Q)) - d(\tilde{c}_0 + \sum_{t=1}^{r} \tilde{c}_t x(S_t \cap Q)) \right| \in O(\epsilon^2 |d|).$$

Adding up over all terms and noting that the sum of $|d|$ over all cut arrays used in the decomposition for one $B^{(z)}$ is $O(1/\epsilon)$, we get the first part of Lemma 11.

Now, we prove the upper bounds on $|l_j|, \sum_j l_j^2$. First note that each $l_j \in O(1/n\epsilon)$, since the sum of the $|d|$ corresponding to all the cut arrays is $O(1/\epsilon)$. If $I(b,\eta) = \emptyset$, then we may set all $l_j = 0$. So, assume $\exists y \in I(b,\eta)$. We claim that in fact $|l(x) - P(x)| \in O(\epsilon)$ for $x \in I(b, 2\eta)$. This is because, the error $\Delta(x) = g_1(x) - g_2(x)$ was bounded by $2^{r+2}|d|\eta^2$ for $x \in I(b,\eta)$; now for $x \in I(b,2\eta)$, $\Delta(x)$ may be bounded above by $2^{r+4}|d|\eta^2$. Thus, $|g_1(x) - g_2(x)| \leq 2^{r+4}|d|\eta^2$ for all $x \in I(b,2\eta)$. Now again, adding up over all terms and noting that $\eta \in O(\epsilon)$, we get that $|l(x) - P(x)| \in O(\epsilon)$ as claimed.

Let $L = \{j : y_j \leq 1/2; l_j > 0\}$. Let $n_0 = \min(|L|, \eta n)$ and denote by J_0 the set of n_0 j 's with the largest values of l_j. Obtain y' from y by making the coordinates in J_0 equal to 1, leaving the other coordinates as in y. This changes each $x(S_t)$ by at most ηn; so y' is still in $I(b, 2\eta)$. Now, this changes $P(\cdot)$ by at most $O(1/n)\min(|L|, \eta n)) \in O(\epsilon)$; so we have that

$$l(y') - l(y) = P(y') - P(y) + (l(y') - P(y')) + (P(y) - l(y)) \in O(\epsilon).$$

But $l(y') - l(y)$ is at least $(1/2)$ the sum of the $\min(|L|, \eta n)$ largest positive l_j among $j \in L$. Thus the sum of the largest $\min(|L|, \eta n)$ l_j is $O(\epsilon)$. Similarly, we may define $L' = \{j : y_j \geq 1/2; l_j > 0\}$ and then modify y to y' by setting to 0 the y_j for the $\mathrm{Min}(|L'|, \eta n)$ $j \in L'$; from this, we get that the sum of the largest $\min(|L'|, \eta n)$ l_j, $j \in L'$ is at most $O(\epsilon)$. Adding, we get that the sum of the largest ηn positive l_j is at most $O(\epsilon)$. By similar argument, we get that the sum of the least ηn l_j is at least $-O(\epsilon)$. So the sum of the ηn largest absolute value l_j is at most $O(\epsilon)$. The largest value of $\sum_j l_j^2$ subject to this condition and the condition that each $|l_j|$ is at most $O(1/n\epsilon)$ is obtained when the top $O(\epsilon^2 n)$ of the $|l_j|$ are $O(1/n\epsilon)$ each and the rest $O(1/n)$.

This completes the proof of Lemma 11.

$\square$

Now we are ready to prove Theorem 1. Let $\mathrm{Max}(F) = \alpha n^r$. Then for each b, the maximum value of the following Integer Program is at most $\alpha + O(\epsilon)$:

$$\mathrm{Max}\ l_0 + l_1 x_1 + l_2 x_2 + \dots l_n x_n$$
$$b_t - 2\eta \leq \frac{1}{n}x(S_t) \leq b_t + 2\eta \text{ for } t = 1, 2, \dots s \quad ; \quad 0 \leq x_j \leq 1 \quad \text{integer.}$$

This implies that

$$\alpha + O(\epsilon) \geq \mathrm{Max}\ l_0 + l_1 x_1 + l_2 x_2 + \dots l_n x_n$$
$$b_t - 2\eta + \frac{s}{n} \leq \frac{1}{n}x(S_t) \leq b_t + 2\eta - \frac{s}{n} \text{ for } t = 1, 2, \dots s \quad ; \quad 0 \leq x_j \leq 1$$

because, for the linear program, there is a basic optimal solution which has at most s fractional variables and setting them to 0 gives us an integer solution whose objective value is at least the linear program value minus $O(s/\epsilon n)$ which is $O(\epsilon)$.

Now we wish to apply Theorem 10. To this end, we note that $||U||_\infty \leq 1/n$; and we may use $M_2 = O(1/\epsilon n)$ and $\alpha_2 = O(1/n)$ in that theorem. Also, we will use $\gamma = O(\epsilon^2)$; note that this satisfies the required lower bound on γ in that theorem. We will also use the fact that s/n is at most $\eta/2$ and $\frac{\eta}{2} \geq 2\sqrt{\gamma}n||U||_\infty$ (the last requires us to choose η not too small; indeed η equal to a large constant times ϵ will do). Thus, we get for the following event $E_{22}(b)$ (for one fixed b) the claimed probability bound (for a suitable choice of $\gamma \in O(\epsilon^2)$)

$$E_{22}(b) : \frac{q}{n}(\alpha - l_0 + O(\epsilon)) > \text{Max} \sum_{j\in Q} l_j x_j$$

$$b_t - \eta \leq \frac{1}{q} x(S_t \cap Q) \leq b_t + \eta \quad \text{for } t = 1, 2, \ldots s \quad ; \quad 0 \leq x_j \leq 1.$$

$$\mathbf{Pr}(E_{22}) \geq 1 - e^{-10\log(1/\epsilon)/\epsilon^2}$$

Applying Lemma 11, we see that $E_{22}(b), E_{15}, E_{29}$ together imply

$$E_{25}(b) : \alpha + O(\epsilon) > \text{Max } \tilde{l}_0 + \sum_{j\in Q} \tilde{l}_j x_j$$

$$b_t - \eta \leq \frac{1}{q} x(S_t \cap Q) \leq b_t + \eta \quad \text{for } t = 1, 2, \ldots s \quad ; \quad 0 \leq x_j \leq 1.$$

The upper bound on objective function of the above Linear Programming also applies to the corresponding Integer Program. Now appealing again to Lemma 11, we get that

$$E_{25}(b) \Longrightarrow \tilde{g}(x) \leq \alpha + O(\epsilon) \; \forall x : x_j \in \{0,1\}, j \in Q, x \in \tilde{I}(b, \eta).$$

Letting,

$$E_{25} : E_{25}(b) \text{ holds for all } b \in \mathcal{A}^s,$$

we then see that since the $\tilde{I}(b, \eta)$ together cover all of $\{0,1\}^q$, under E_{25}, we have that

$$\tilde{g}(x) \leq \alpha + O(\epsilon) \; \forall x : x_j \in \{0,1\}, j \in Q,$$

and also we have that

$$\mathbf{Pr}(E_{25}) \geq 1 - (\text{ number of } b \text{ 's}) e^{-10\log(1/\epsilon)/\epsilon^2} \geq 99/100.$$

To complete the proof of (34) (and hence the Theorem), we only need to prove now that with high probability,

$$\text{Max}_{x_j \in \{0,1\}, j\in V} \quad g(x) \leq \text{Max}_{x_j \in \{0.1\}, j \in Q} \quad \tilde{g}(x) + O(\epsilon).$$

This is the easy part and we will only sketch the routine proof. Suppose z attains

$$\text{Max}_{x_j \in \{0,1\}, j\in V}\, g(x).$$

Then, under E_{15}, arguing as in Lemma 12, we see that

$$|g_1(z) - \tilde{g}_1(z)| \in O(\epsilon^2).$$

Now adding up over all cut arrays in the decomposition and noting again that the sum of the $|d|$'s is $O(1/\epsilon)$, it follows that $\tilde{g}(z) \geq g(z) - O(\epsilon)$ proving this part.

□

8 Lower Bound on Number of Cut Arrays Needed

In this section we show that the $c(r)/\epsilon^2$ upper estimate for the number of cut arrays in Theorem 4 is tight (up to the dependence on r), even if we restrict our attention to $\{-1,1\}$-arrays A, and even if we only require that the sum of the cut arrays D will satisfy (6). Throughout the subsection we assume, whenever this is needed, that ϵ is sufficiently small as a function of r. We also omit all floor and ceiling signs whenever these are not crucial, to simplify the presentation. Note that if we only wish to satisfy (6) in Theorem 4, then its proof implies that $1/\epsilon^2$ cut arrays suffice, as the extra 4 term appears because of the need to get an efficient algorithm.

The L_1-norm of an array $A : V_1 \times V_2 \cdots \times V_r \mapsto R$ is given by

$$||A||_1 = \sum_{(i_1,i_2,\ldots,i_r) \in V_1 \times V_2 \cdots \times V_r} |A(i_1, i_2, \ldots, i_r)|.$$

The following lemma supplies a lower bound for the cut-norm of an array in terms of its L_1-norm. The proof is based on the method of [3].

Lemma 13. *Let $A : V_1 \times V_2 \cdots \times V_r \mapsto R$ be an array. Then its cut norm satisfies*

$$||A||_C \geq \frac{||A||_1}{2 \cdot 8^{(r-1)/2} \prod_{j=2}^{r} |V_j|^{1/2}}.$$

The proof (following the ideas of [3]) uses a result of Szarek. Let $c_1, c_2, \ldots, c_n$ be a set of n reals, let $\delta_1, \ldots, \delta_n$ be independent, identically distributed random variables, each distributed uniformly on $\{-1,1\}$, and define $X = \sum_i \delta_i c_i$.

Lemma 14. (Szarek [17]) *In the above notation,*

$$E(|X|) \geq 2^{-1/2}(c_1^2 + \ldots + c_n^2)^{1/2} \quad (\geq \frac{|c_1| + \ldots + |c_n|}{\sqrt{2n}}.)$$

Corollary 15. *Let $c_1, \ldots, c_n$ be reals, and let S be a random subset of $\{1, 2, \ldots, n\}$ taken uniformly among all 2^n subsets. Let Y be the random variable $Y = \sum_{i \in S} c_i$. Then*

$$E(|Y|) = \frac{\sum_{S \subset \{1,\ldots n\}} |\sum_{i \in S} c_i|}{2^n} \geq \frac{\sum_i |c_i|}{\sqrt{8n}}$$

Proof For every vector $\delta = (\delta_1, \ldots, \delta_n) \in \{-1, 1\}^n$ define $S_\delta = \{i : \delta_i = 1\}$ and $S'_\delta = \{i : \delta_i = -1\}$. Then, by the triangle inequality

$$|\sum_{i \in S_\delta} c_i| + |\sum_{i \in S'_\delta} c_i| \geq |\sum_i \delta_i c_i|.$$

As δ ranges over all 2^n members of $\{-1, 1\}^n$, S_δ, as well as S'_δ range over all 2^n subsets of $\{1, 2, \ldots, n\}$ implying that $2E(|Y|) \geq E(|X|)$, where X is as above. The result now follows from Lemma 14. $\Box$

Proof of Lemma 13. We prove, by induction on t, that for every $0 \leq t \leq r$ there are subsets $S_{r-t+1} \subset V_{r-t+1} \ldots S_r \subset V_r$ such that

$$\sum_{i_1 \in V_1} \cdots \sum_{i_{r-t} \in V_{r-t}} |\sum_{i_{r-t+1} \in S_{r-t+1}} \cdots \sum_{i_r \in S_r} A(i_1, i_2, \ldots, i_r)| \geq \frac{||A||_1}{8^{t/2} \prod_{j=r-t+1}^r |V_j|^{1/2}}. \tag{35}$$

For $t = 0$ there is nothing to prove. Assuming the assertion holds for $t - 1 < r$, we prove it for t. For each $(r-t)$-tuple $i_1, i_2, \ldots, i_{r-t}$ and each $i \in V_{r-t+1}$ define

$$c_i = c_i(i_1, i_2, \ldots, i_{r-t}) = \sum_{i_{r-t+2} \in S_{r-t+2}} \cdots \sum_{i_r \in S_r} A(i_1, i_2, \ldots, i_{t-r}, i, i_{t-r+2}, \ldots, i_r),$$

and apply Corollary 15 with $n = |V_{r-t+1}|$. Summing the resulting inequalities for all $(i_1, \ldots, i_{r-t}) \in V_1 \times \cdots \times V_{r-t}$ we conclude that the average (over $S_{r-t+1} \subset V_{r-t+1}$) of the sum

$$\sum_{i_1 \in V_1} \cdots \sum_{i_{r-t} \in V_{r-t}} |\sum_{i_{r-t+1} \in S_{r-t+1}} \cdots \sum_{i_r \in S_r} A(i_1, i_2, \ldots, i_r)|$$

is at least

$$\frac{1}{\sqrt{8|V_{r-t+1}|}} \frac{||A||_1}{8^{(t-1)/2} \prod_{j=r-t+2}^r |V_j|^{1/2}} = \frac{||A||_1}{8^{t/2} \prod_{j=r-t+1}^r |V_j|^{1/2}}.$$

Therefore, there is a set $S_{r-t+1} \subset V_{r-t+1}$ for which (35) holds, showing that it indeed holds for all $t \leq r$.

In particular, for $t = r - 1$ there are sets $S_2 \subset V_2, \ldots, S_r \subset V_r$ such that

$$\sum_{i_1 \in V_1} |\sum_{i_2 \in S_2} \cdots \sum_{i_r \in V_r} A(i_1, i_2, \ldots, i_r)| \geq \frac{||A||_1}{8^{(r-1)/2} \prod_{j=2}^r |V_j|^{1/2}}. \tag{36}$$

Fixing such sets S_i, either the contribution of the positive terms $\sum_{i_2 \in S_2} \cdots \sum_{i_r \in V_r} A(i_1, i_2, \ldots, i_r)$ gives at least half of (36), or the contribution of the absolute values of the negative terms gives at least half the sum. In each case we can define S_1 as the set of those $i_1 \in V_1$ that correspond to those contributing terms and conclude that

$$||A||_C \geq |\sum_{i_1 \in S_1} \cdots \sum_{i_r \in S_r} A(i_1, \ldots, i_r)|$$

$$\geq \frac{||A||_1}{2 \cdot 8^{(r-1)/2} \prod_{j=2}^r |V_j|^{1/2}}.$$

This completes the proof. □

From now on we restrict our attention in this subsection to arrays $A : V_1 \times V_2 \times \cdots \times V_r \mapsto \{-1, 1\}$ where $|V_i| = n$ for all i. We need the following simple fact.

Lemma 16. *There exists a family $\mathcal{F}$ of r-dimensional arrays, each mapping $V_1 \times V_2 \times \cdots \times V_r$, where $|V_i| = n$ for each i, into $\{-1, 1\}$ such that $|\mathcal{F}| \geq 2^{n^r/2}$ and for each two distinct members $A, B \in \mathcal{F}$, $||A - B||_1 > \frac{n^r}{5}$.*

Proof Let $H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$ be the binary entropy function. By the Gilbert-Varshamov bound (see, e.g., [16]), for every (large) m there are at least $2^{(1-H(1/10))m}$ ($> 2^{m/2}$) vectors of length m over $\{-1, 1\}$, where the Hamming distance between each pair exceeds $m/10$. Taking $m = n^r$ and viewing these vectors as arrays mapping $V_1 \times \cdots \times V_r$ to $\{-1, 1\}$, the desired result follows, as the difference between any two distinct arrays in the collection will have more than $n^r/10$ nonzero entries, each of which is either 2 or -2. □

We can now prove the main result of this subsection.

Theorem 17. *For every fixed dimension $r \geq 2$ there exists some $c(r) > 0$ so that for every $\epsilon > 0$ there are $n, N = n^r$ and an r-dimensional array $A : V_1 \times \cdots \times V_r \mapsto \{-1, 1\}$, where $|V_i| = n$ for all i, such that for every array D which is the sum of less than $c(r)/\epsilon^2$ cut arrays,*

$$||A - D||_C > \epsilon n^r \ (\ = \epsilon\sqrt{N}||A||_F)$$

Proof We prove the theorem for all ϵ which is sufficiently small as a function of r, and with $c(r) = \frac{1}{4r \cdot 40^2 \cdot 8^{r-1}}$. Clearly this implies the result for all ϵ (with a possibly smaller $c = c(r)$). Define

$$n = \frac{1}{8 \cdot (40)^{2/(r-1)} \epsilon^{2/(r-1)}},$$

and note that $N = n^r < 1/(2\epsilon^4)$. By Lemma 16 there is a family $\mathcal{F}$ of $2^{n^r/2}$ arrays $A : V_1 \times V_2 \times \cdots \times V_r \mapsto \{-1, 1\}$ such that for every two distinct members $A, B \in \mathcal{F}$, $||A - B||_1 > N/5$. By Lemma 13 this implies that for every such A, B,

$$||A - B||_C \geq \frac{||A - B||_1}{2 \cdot 8^{(r-1)/2} n^{(r-1)/2}} > \frac{n^{(r+1)/2}}{10 \cdot 8^{(r-1)/2}} = 4\epsilon n^r, \tag{37}$$

where the last equality follows from the definition of n.

Therefore, $\mathcal{F}$ is a large set of arrays, so that the cut-distance between any pair of them is large. To complete the proof we show that at least one member of $\mathcal{F}$ cannot be approximated well (in the cut metric) by a sum of a small number of cut arrays. To do so, suppose that for each member A of $\mathcal{F}$ there is an array D which is a sum of at most t cut arrays, such that $||A - D||_C \leq \epsilon n^r$. Call a cut-array ϵ-nice if it is an array of the form $CUT(S_1, S_2, \ldots, S_r; d)$ where d is an integral multiple of ϵ/t. An obvious rounding procedure implies that for each member of $\mathcal{F}$ there is an array D which is the sum of at most t ϵ-nice cut arrays, such that $||A - D||_C < 2\epsilon n^r$.

We next prove an upper bound for the total possible number of such arrays D. Note, first, that as $n^r < 1/(2\epsilon^4)$, the absolute value of no entry of such a D can exceed $1 + 1/\epsilon^3 < 2/\epsilon^3$ (since otherwise the cut-norm of $A - D$ would exceed $2\epsilon n^r$ simply by considering a single entry). As each entry of D is also an integral multiple of ϵ/t it follows that there are at most $4t/\epsilon^4$ possibilities for each such entry. There are at most 2^{nr} possibilities for choosing the sets $S_1, \ldots, S_r$ in each cut array $CUT(S_1, \ldots, S_r; d)$, and as D is the sum of t such arrays there are at most 2^{nrt} possibilities for choosing the defining sets of all of them. Once these are chosen, we have to choose the densities d of these arrays. Each of those is an integral multiple of ϵ/t, but the trouble is that its absolute value may be large (as there may be cancellations between them, while forming D). It is thus better to bound the number of possibilities of all these densities as follows. Let $d_1, \ldots, d_t$ be the densities. Since we have already chosen all sets S_i in all the cut arrays whose sum is D, we can express each entry of D as a sum of a subset of the densities d_i. At most t of the characteristic vectors of these subsets span all the characteristic vectors of all other subsets we have, and thus if we are given the values of D in these entries, we can solve for all other entries of D. There are at most n^{rt} ways to choose t entries of D, and then there are at most $(4t/\epsilon)^t$ possibilities for the values of D in these entries (as each entry is an integral multiple of ϵ/t whose absolute value does not exceed $2/\epsilon^3$.) Therefore, the total number of possible arrays D is at most

$$n^{rt}(\frac{4t}{\epsilon})^t 2^{nrt}.$$

Each member of $\mathcal{F}$ is within cut-distance smaller than $2\epsilon n^r$ from at least one of these arrays D, and the cut-distance between any two distinct members of $\mathcal{F}$ exceeds $4\epsilon n^r$, by (37). It thus follows that the number of arrays D is at least as large as $\mathcal{F}$, implying that

$$\log|\mathcal{F}| = \frac{n^r}{2} \leq rt\log n + t\log(4t/\epsilon) + nrt < 2trn,$$

where here we used the fact that n is much bigger than $\log n + \log(4t/\epsilon)$. The last inequality implies that

$$t \geq \frac{n^{r-1}}{4r} = \frac{1}{4r \cdot 40^2 \cdot 8^{r-1}\epsilon^2},$$

completing the proof. □

9 Acknowledgments

We thank Alan Frieze, Oded Goldreich and Claire Kenyon for helpful comments. We thank the referees for a thorough and painstaking reading of the paper and many excellent suggestions.

References

[1] G. Andersson and L. Engebretsen, *Property testers for dense constraint satisfaction programs on finite domains*, Random Structures and Algorithms 21 (2002), 14-32.

[2] G. Andersson and L. Engebretsen, *Sampling methods applied to dense instances of non-Boolean optimization problems*, Proceedings of the 2nd International Workshop on Randomization and Approximation Techniques in Computer Science, LNCS 1017, 357-368, Barcelona, October 1998, Springer-Verlag.

[3] N. Alon, B. Doerr, T. Luczak and T. Schoen, On the discrepancy of combinatorial rectangles, Random Structures and Algorithms, to appear.

[4] N. Alon, W. Fernandez-de-la-Vega, R. Kannan and M. Karpinski, Random Sampling and approximation of MAX-CSP problems, Proc. of the 34^{th} ACM STOC, ACM Press (2002), 232-239.

[5] N. Alon and A. Shapira, Testing satisfiability, Proc. of the 13^{th} Annual ACM-SIAM SODA, ACM Press (2002), 645-654.

[6] S.Arora, D.Karger and M.Karpinski, *Polynomial time approximation schemes for dense instances of NP-hard problems*, Proceedings of the 27th Annual ACM Symposium on Theory of Computing (1995) 284-293; the full paper appeared in J. Comp. System. Sci. **58** (1999), 193-210.

[7] K. Azuma, Weighted Sum of Certain Dependent random Variables, Tôhoku Math.J. 3 (1967), 357-367.

[8] A. Czumaj, C. Sohler, *Testing Hypergraph Coloring*, Proc. 28th ICALP'01, LNCS Vol. 2076, Springer, 2001, 493-505.

[9] W.Fernandez-de-la-Vega, *MAX-CUT has a Randomized Approximation Scheme in Dense Graphs*, Random Structures and Algorithms 8 (1996) 187-199.

[10] A.M.Frieze and R.Kannan, *The Regularity Lemma and approximation schemes for dense problems*, Proceedings of the 37th Annual IEEE Symposium on Foundations of Computing, (1996) 12-20.

[11] A. M. Frieze and R. Kannan *Quick Approximation to matrices and applications*, Combinatorica **19** (2) (1999) pp 175-200.

[12] M.X.Goemans and D.P.Williamson, *.878-Approximation algorithms for MAX CUT and MAX 2SAT*, Proceedings of the 26th Annual ACM Symposium on Theory of Computing, ACM Press (1994), 422-431.

[13] O.Goldreich, S.Goldwasser and D.Ron, *Property testing and its connection to learning and approximation*, Proc. 37th IEEE FOCS 96, 339-348; the full paper appeared in J. ACM **45** (1998), 653-750.

[14] W.Hoeffding, *Probability inequalities for sums of bounded random variables*, Journal of the American Statistical Association 58 (1963) 13-30.

[15] S. Khanna, M. Sudan and D. Williamson, *A complete classification of the approximability of maximization problems derived from Boolean constraint satisfaction*, Proc. 29th ACM STOC (1997), 11-20.

[16] F.J. MacWilliams and N.J.A. Sloane, *The Theory of Error-Correcting Codes*, North Holland, Amsterdam, 1977.

[17] S. J. Szarek, On the best constants in the Khinchin Inequality, Studia Math. 58 (1976), 197-208.

[18] V.N. Vapnik and A.Y. Chervonenkis. On the uniform convergence of relative frequencies of events to their probabilities, *Theory of Probability and its Applications*, 16(2) (1971), 264–280.