Spectral Graph Theory Lecture 18

Linear Sized Sparsifiers

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Disclaimer

These notes are not necessarily an accurate representation of what happened in class. The notes written before class say what I think I should say. I sometimes edit the notes after class to make them way what I wish I had said.

There may be small mistakes, so I recommend that you check any mathematically precise statement before using it in your own work.

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18.1 Acknowledgement

I am grateful to David Williamson for catching mistakes in these notes and for suggesting how to correct them.

18.2 Overview

In this lecture, we will prove a slight simplification of the main result of [**?**, **?**]. This will tell us that every graph with n vertices has an ϵ-approximation with approximately $4\epsilon^{-2}n$ edges. To translate this into a relation between approximation quality and average degree, note that such a graph has average degree $d_{ave} = 8\epsilon^{-2}$. So,

$$\epsilon \approx \frac{2\sqrt{2}}{d},$$

which is about twice what you would get from a Ramanujan graph. Interestingly, this result even works for average degree just a little bit more than 1.

18.3 Turning edges into vectors

In the last lecture, we considered the Laplacian matrix of a graph G times the square root of the pseudoinverse on either side. That is,

$$\boldsymbol{L}_G^{+/2}\left(\sum_{(a,b)\in E} w_{a,b}\boldsymbol{L}_{(a,b)}\right)\boldsymbol{L}_G^{+/2}.$$

Today, it will be convenient to view this as a sum of outer products of vectors. Set

$$\boldsymbol{v}_{(a,b)} = \sqrt{w_{a,b}}\boldsymbol{L}_G^{+/2}(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b).$$

Then,

$$\boldsymbol{L}_G^{+/2}\left(\sum_{(a,b)\in E} w_{a,b}\boldsymbol{L}_{(a,b)}\right)\boldsymbol{L}_G^{+/2} = \sum_{(a,b)\in E}\boldsymbol{v}_{(a,b)}\boldsymbol{v}_{(a,b)}^T.$$

The problem of sparsification is then the problem of finding a small subset of these vectors, $S \subseteq E$, along with scaling factors, $c : S \to \mathbb{R}$, so that

$$\sum_{(a,b)\in S} c_{a,b}\boldsymbol{v}_{(a,b)}\boldsymbol{v}_{(a,b)}^T \approx_\epsilon \sum_{(a,b)\in E}\boldsymbol{v}_{(a,b)}\boldsymbol{v}_{(a,b)}^T.$$

If we project onto the span of the Laplacian, then the sum of the outer products of vectors $\boldsymbol{v}_{(a,b)}$ becomes the identity, and our goal is to find a set S and scaling factors $c_{a,b}$ so that

$$(1-\epsilon)\boldsymbol{I} \preccurlyeq \sum_{(a,b)\in S} c_{a,b}\boldsymbol{v}_{(a,b)}\boldsymbol{v}_{(a,b)}^T \preccurlyeq (1+\epsilon)\boldsymbol{I}.$$

That is, so that all the eigenvalues of the matrix in the middle lie between $(1-\epsilon)$ and $(1+\epsilon)$.

18.4 The main theorem

Theorem 18.4.1. *Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ be vectors in $\mathbb{R}^n$ so that*

$$\sum_i \boldsymbol{v}_i\boldsymbol{v}_i^T = \boldsymbol{I}.$$

Then, for every $\epsilon > 0$ there exists a set S along with scaling factors c_i so that

$$(1-\epsilon)^2\boldsymbol{I} \preccurlyeq \sum_{i\in S} c_i\boldsymbol{v}_i\boldsymbol{v}_i^T \preccurlyeq (1+\epsilon)^2\boldsymbol{I},$$

and

$$|S| \leq \left\lceil n/\epsilon^2 \right\rceil.$$

The condition that the sum of the outer products of the vectors sums to the identity has a name, *isotropic position*. I now mention one important property of vectors in isotropic position

Lemma 18.4.2. *Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ be vectors in isotropic position. Then, for every matrix $\boldsymbol{M}$,*

$$\sum_i \boldsymbol{v}_i^T \boldsymbol{M} \boldsymbol{v}_i = \mathrm{Tr}\left(\boldsymbol{M}\right).$$

Proof. We have

$$\boldsymbol{v}^T \boldsymbol{M} \boldsymbol{v} = \mathrm{Tr}\left(\boldsymbol{v}\boldsymbol{v}^T \boldsymbol{M}\right),$$

so

$$\sum_i \boldsymbol{v}_i^T \boldsymbol{M} \boldsymbol{v}_i = \sum_i \mathrm{Tr}\left(\boldsymbol{v}_i \boldsymbol{v}_i^T \boldsymbol{M}\right) = \mathrm{Tr}\left(\left(\sum_i \boldsymbol{v}_i \boldsymbol{v}_i^T\right) \boldsymbol{M}\right) = \mathrm{Tr}\left(\boldsymbol{I}\boldsymbol{M}\right) = \mathrm{Tr}\left(\boldsymbol{M}\right).$$

□

Today, we will prove that we can find a set of $6n$ vectors for which all eigenvalues lie between $1n$ and $13n$. If you divide all scaling factors by $\sqrt{13}n$, this puts the eigenvalues between $1/\sqrt{13}$ and $\sqrt{13}$. You can tighten the argument to prove Theorem 18.3.1.

We will prove this theorem by an iterative argument in which we choose one vector at a time to add to the set S. We will set the scaling factor of a vector when we add it to S. It is possible that we will add a vector to S more than once, in which case we will increase its scaling factor each time. Throughout the argument we will maintain the invariant that the eigenvalues of the scaled sum of outer produces is in the interval $[l, u]$, where l and u are quantities that will change with each addition to S. At the start of the algorithm, when S is empty, we will have

$$l_0 = -n \quad \text{and} \quad u_0 = n.$$

Every time we add a vector to S, we increase l by δ_L and u by δ_U, where

$$\delta_L = 1/3 \quad \text{and} \quad \delta_U = 2.$$

After we have done this $6n$ times, we will have $l = n$ and $u = 13n$.

18.5 Rank-1 updates

We will need to understand what happens to a matrix when we add the outer product of a vector.

Theorem 18.5.1 (Sherman-Morrison)**.** *Let $\boldsymbol{A}$ be a nonsingular symmetric matrix and let $\boldsymbol{v}$ be a vector and let c be a real number. Then,*

$$(\boldsymbol{A} - c\boldsymbol{v}\boldsymbol{v}^T)^{-1} = \boldsymbol{A}^{-1} + c\frac{\boldsymbol{A}^{-1}\boldsymbol{v}\boldsymbol{v}^T\boldsymbol{A}^{-1}}{1 - c\boldsymbol{v}^T\boldsymbol{A}^{-1}\boldsymbol{v}}.$$

Proof. The easiest way to prove this is to multiply it out, gathering $\boldsymbol{v}^T \boldsymbol{A}^{-1} \boldsymbol{v}$ terms into scalars:

$$
\begin{aligned}
(\boldsymbol{A} - c\boldsymbol{v}\boldsymbol{v}^T)\left(\boldsymbol{A}^{-1} + c\frac{\boldsymbol{A}^{-1}\boldsymbol{v}\boldsymbol{v}^T\boldsymbol{A}^{-1}}{1 - c\boldsymbol{v}^T\boldsymbol{A}^{-1}\boldsymbol{v}}\right) &= \boldsymbol{I} - c\boldsymbol{v}\boldsymbol{v}^T\boldsymbol{A}^{-1} + c\frac{\boldsymbol{v}\boldsymbol{v}^T\boldsymbol{A}^{-1}}{1 - c\boldsymbol{v}^T\boldsymbol{A}^{-1}\boldsymbol{v}} - c^2\frac{\boldsymbol{v}\boldsymbol{v}^T\boldsymbol{A}^{-1}\boldsymbol{v}\boldsymbol{v}^T\boldsymbol{A}^{-1}}{1 - c\boldsymbol{v}^T\boldsymbol{A}^{-1}\boldsymbol{v}} \\
&= \boldsymbol{I} - c\boldsymbol{v}\boldsymbol{v}^T\boldsymbol{A}^{-1}\left(1 - \frac{1}{1 - c\boldsymbol{v}^T\boldsymbol{A}^{-1}\boldsymbol{v}} + \frac{c\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v}}{1 - c\boldsymbol{v}^T\boldsymbol{A}^{-1}\boldsymbol{v}}\right) \\
&= \boldsymbol{I}.
\end{aligned}
$$

□

18.6 Barrier Function Arguments

To prove the main theorem we need a good way to measure progress. We would like to keep all the eigenvalues of the matrix we have constructed any any point to lie in a nice range. But, more than that, we need them to be nicely distributed within this range. To enforce this, we need to measure how close the eigenvalues are to the limits.

Let $\boldsymbol{A}$ be a symmetric matrix with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$. If u is larger than all of the eigenvalues of $\boldsymbol{A}$, then we call u an upper bound on $\boldsymbol{A}$. To make this notion quantitive, we define the *upper barrier function*

$$\Phi^u(\boldsymbol{A}) = \sum_i \frac{1}{u - \lambda_i}.$$

This is positive for all upper bounds u, goes to infinity as u approaches the largest eigenvalue, decreases as u grows, and is convex for $u > \lambda_n$. In particular, we will use

$$\Phi^{u+\delta}(\boldsymbol{A}) < \Phi^u(\boldsymbol{A}), \qquad \text{for } \delta > 0. \tag{18.1}$$

Also, observe that

$$\lambda_n \leq u - 1/\Phi^u(\boldsymbol{A}). \tag{18.2}$$

We will exploit the following formula for the upper barrier function:

$$\Phi^u(\boldsymbol{A}) = \mathrm{Tr}\left((u\boldsymbol{I} - \boldsymbol{A})^{-1}\right).$$

For a lower bound on the eigenvalues l, we will define an analogous lower barrier function

$$\Phi_l(\boldsymbol{A}) = \sum_i \frac{1}{\lambda_i - l} = \mathrm{Tr}\left((\boldsymbol{A} - l\boldsymbol{I})^{-1}\right).$$

This is positive whenever l is smaller than all the eigenvalues, goes to infinity as l approaches the smallest eigenvalue, and decreases as l becomes smaller. In particular,

$$l + 1/\Phi_l(\boldsymbol{A}) \leq \lambda_1. \tag{18.3}$$

The analog of (18.1) is the following.

Claim 18.6.1. *Let l be a lower bound on $\boldsymbol{A}$ and let $\delta < 1/\Phi_l(\boldsymbol{A})$. Then,*

$$\Phi_{l+\delta}(\boldsymbol{A}) \leq \frac{1}{1/\Phi_l(\boldsymbol{A}) - \delta}.$$

Proof. This will really be more of a sketch of a proof than a proof. But, once you understand what it going on, it would not be too hard to write a proof. Consider the derivative of the barrier function in l:

$$\frac{\partial}{\partial l}\Phi_l(\boldsymbol{A}) = \sum_{i=1}^{n} \frac{\partial}{\partial l}\frac{1}{\lambda_i - l} = \sum_{i=1}^{n}\left(\frac{1}{\lambda_i - l}\right)^2.$$

So,

$$\frac{\partial}{\partial l}\Phi_l(\boldsymbol{A}) \leq \Phi_l(\boldsymbol{A})^2,$$

with equality only when $\lambda_2, \ldots, \lambda_n$ are infinite. In the case where equality holds, we have

$$\Phi_l(\boldsymbol{A}) = \frac{1}{\lambda_1 - l},$$

and

$$\begin{aligned}\Phi_{l+\delta}(\boldsymbol{A}) &= \frac{1}{\lambda_1 - l - \delta} \\ &= \frac{1}{1/\Phi_l(\boldsymbol{A}) - \delta}.\end{aligned}$$

In every other case, $\Phi_{l+\delta}(\boldsymbol{A})$ is smaller. □

Initially, we will have

$$\Phi_{l_0}(0) = \Phi_{-n}(0) = 1 \quad \text{and} \quad \Phi^{u_0}(0) = \Phi^{n}(0) = 1.$$

18.7 Barrier Function Updates

The most important thing to understand about the barrier functions is how they change when we add a vector to S. The Sherman-Morrison theorem tells us that happens when we change $\boldsymbol{A}$ to $\boldsymbol{A} + c\boldsymbol{v}\boldsymbol{v}^T$:

$$\begin{aligned}\Phi^u(\boldsymbol{A} + c\boldsymbol{v}\boldsymbol{v}^T) &= \operatorname{Tr}\left((u\boldsymbol{I} - \boldsymbol{A} - c\boldsymbol{v}\boldsymbol{v}^T)^{-1}\right) \\ &= \operatorname{Tr}\left((u\boldsymbol{I} - \boldsymbol{A})^{-1}\right) + c\frac{\operatorname{Tr}\left((u\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{v}\boldsymbol{v}^T(u\boldsymbol{I} - \boldsymbol{A})^{-1}\right)}{1 - c\boldsymbol{v}^T(u\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{v}} \\ &= \Phi^u(\boldsymbol{A}) + c\frac{\operatorname{Tr}\left(\boldsymbol{v}^T(u\boldsymbol{I} - \boldsymbol{A})^{-1}(u\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{v}\right)}{1 - c\boldsymbol{v}^T(u\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{v}} \\ &= \Phi^u(\boldsymbol{A}) + c\frac{\boldsymbol{v}^T(u\boldsymbol{I} - \boldsymbol{A})^{-2}\boldsymbol{v}}{1 - c\boldsymbol{v}^T(u\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{v}}.\end{aligned}$$

This increases the upper barrier function, and we would like to counteract this increase by increasing u at the same time. If we advance u to $\hat{u} = u + \delta_U$, then we find

$$\Phi^{u+\delta_U}(\boldsymbol{A} + c\boldsymbol{v}\boldsymbol{v}^T) = \Phi^{u+\delta_U}(\boldsymbol{A}) + c\frac{\boldsymbol{v}^T(\hat{u}\boldsymbol{I} - \boldsymbol{A})^{-2}\boldsymbol{v}}{1 - c\boldsymbol{v}^T(\hat{u}\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{v}}$$

$$= \Phi^{u}(\boldsymbol{A}) - \left(\Phi^{u}(\boldsymbol{A}) - \Phi^{u+\delta_U}(\boldsymbol{A})\right) + \frac{\boldsymbol{v}^T(\hat{u}\boldsymbol{I} - \boldsymbol{A})^{-2}\boldsymbol{v}}{1/c - \boldsymbol{v}^T(\hat{u}\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{v}}.$$

We would like for this to be less than $\Phi^u(\boldsymbol{A})$. If we commit to how much we are going to increase u, then this gives an upper bound on how large c can be. We want

$$\left(\Phi^{u}(\boldsymbol{A}) - \Phi^{u+\delta_U}(\boldsymbol{A})\right) \geq \frac{\boldsymbol{v}^T(\hat{u}\boldsymbol{I} - \boldsymbol{A})^{-2}\boldsymbol{v}}{1/c - \boldsymbol{v}^T(\hat{u}\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{v}},$$

which is equivalent to

$$\frac{1}{c} \geq \frac{\boldsymbol{v}^T(\hat{u}\boldsymbol{I} - \boldsymbol{A})^{-2}\boldsymbol{v}}{(\Phi^{u}(\boldsymbol{A}) - \Phi^{u+\delta_U}(\boldsymbol{A}))} + \boldsymbol{v}^T(\hat{u}\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{v}.$$

Define

$$\boldsymbol{U}_{\boldsymbol{A}} = \frac{((u+\delta_u)\boldsymbol{I} - \boldsymbol{A})^{-2}}{(\Phi^{u}(\boldsymbol{A}) - \Phi^{u+\delta_U}(\boldsymbol{A}))} + ((u+\delta_u)\boldsymbol{I} - \boldsymbol{A})^{-1}.$$

We have established a clean condition for when we can add $c\boldsymbol{v}\boldsymbol{v}^T$ to S and increase u by δ_U without increasing the upper barrier function.

Lemma 18.7.1. *If*

$$\frac{1}{c} \geq \boldsymbol{v}^T\boldsymbol{U}_{\boldsymbol{A}}\boldsymbol{v},$$

then

$$\Phi^{u+\delta_U}(\boldsymbol{A} + c\boldsymbol{v}\boldsymbol{v}^T) \leq \Phi^{u}(\boldsymbol{A}).$$

The miracle in the above formula is that the condition in the lemma just involves the vector $\boldsymbol{v}$ as the argument of a quadratic form.

We also require the following analog for the lower barrier function. The difference is that increasing l by setting $\hat{l} = l + \delta_L$ increases the barrier function, and adding a vector decreases it.

Lemma 18.7.2. *Define*

$$\boldsymbol{L}_{\boldsymbol{A}} = \frac{(\boldsymbol{A} - \hat{l}\boldsymbol{I})^{-2}}{(\Phi_{l+\delta_L}(\boldsymbol{A}) - \Phi_{l}(\boldsymbol{A}))} - (\boldsymbol{A} - \hat{l}\boldsymbol{I})^{-1}.$$

If

$$\frac{1}{c} \leq \boldsymbol{v}^T\boldsymbol{L}_{\boldsymbol{A}}\boldsymbol{v},$$

then

$$\Phi_{l+\delta_L}(\boldsymbol{A} + c\boldsymbol{v}\boldsymbol{v}^T) \leq \Phi_{l}(\boldsymbol{A}).$$

If we fix the vector $\boldsymbol{v}$ and an increment δ_L, then this gives a lower bound on the scaling factor by which we need to multiply it for the lower barrier function not to increase.

18.8 The inductive argument

It remains to show that there exits a vector $\boldsymbol{v}$ and a scaling factor c so that

$$\Phi^{u+\delta_U}(\boldsymbol{A} + c\boldsymbol{v}\boldsymbol{v}^T) \leq \Phi^u(\boldsymbol{A}) \quad \text{and} \quad \Phi_{l+\delta_L}(\boldsymbol{A} + c\boldsymbol{v}\boldsymbol{v}^T) \leq \Phi_l(\boldsymbol{A}).$$

That is, we need to show that there is a vector $\boldsymbol{v}_i$ so that

$$\boldsymbol{v}_i^T \boldsymbol{U}_{\boldsymbol{A}} \boldsymbol{v}_i \leq \boldsymbol{v}_i^T \boldsymbol{L}_{\boldsymbol{A}} \boldsymbol{v}_i.$$

Once we know this, we can set c so that

$$\boldsymbol{v}_i^T \boldsymbol{U}_{\boldsymbol{A}} \boldsymbol{v}_i \leq \frac{1}{c} \leq \boldsymbol{v}_i^T \boldsymbol{L}_{\boldsymbol{A}} \boldsymbol{v}_i.$$

Lemma 18.8.1.

$$\sum_i \boldsymbol{v}_i^T \boldsymbol{U}_{\boldsymbol{A}} \boldsymbol{v}_i \leq \frac{1}{\delta_U} + \Phi_u(\boldsymbol{A}).$$

Proof. By Lemma 18.3.2, we know

$$\sum_i \boldsymbol{v}_i^T \boldsymbol{U}_{\boldsymbol{A}} \boldsymbol{v}_i = \operatorname{Tr}\left(\boldsymbol{U}_{\boldsymbol{A}}\right).$$

To bound this, we break it into two parts

$$\frac{\operatorname{Tr}\left((\hat{u}\boldsymbol{I} - \boldsymbol{A})^{-2}\right)}{(\Phi^u(\boldsymbol{A}) - \Phi^{u+\delta_U}(\boldsymbol{A}))}$$

and

$$\operatorname{Tr}\left((\hat{u}\boldsymbol{I} - \boldsymbol{A})^{-1}\right).$$

The second term is easiest

$$\operatorname{Tr}\left((\hat{u}\boldsymbol{I} - \boldsymbol{A})^{-1}\right) = \Phi^{u+\delta}(\boldsymbol{A}) \leq \Phi^u(\boldsymbol{A}).$$

To bound the first term, consider the derivative of the barrier function with respect to u:

$$\frac{\partial}{\partial u}\Phi^u(\boldsymbol{A}) = \frac{\partial}{\partial u}\sum_i \frac{1}{u - \lambda_i} = -\sum_i \left(\frac{1}{u - \lambda_i}\right)^2 = -\operatorname{Tr}\left(u\boldsymbol{I} - \boldsymbol{A}\right)^{-2}.$$

As $\Phi^u(\boldsymbol{A})$ is convex in u, we may conclude that

$$\Phi^u(\boldsymbol{A}) - \Phi^{u+\delta_U}(\boldsymbol{A}) \geq -\delta_U \frac{\partial}{\partial u}\Phi^{u+\delta_u}(\boldsymbol{A}) = \delta_U \operatorname{Tr}\left(u\boldsymbol{I} - \boldsymbol{A}\right)^{-2}.$$

□

The analysis for the lower barrier is similar, but the second term is slightly more complicated.

Lemma 18.8.2.

$$\sum_i \boldsymbol{v}_i^T \boldsymbol{L}_{\boldsymbol{A}} \boldsymbol{v}_i \geq \frac{1}{\delta_L} - \frac{1}{1/\Phi_l(\boldsymbol{A}) - \delta_L}.$$

Proof. As before, we bound

$$\frac{\mathrm{Tr}\left((\boldsymbol{A} - (l + \delta_L \boldsymbol{I}))^{-2} \right)}{\Phi_{l+\delta_L}(\boldsymbol{A}) - \Phi_l(\boldsymbol{A})}$$

by recalling that

$$\frac{\partial}{\partial l} \Phi_l(\boldsymbol{A}) = \mathrm{Tr}\left(\boldsymbol{A} - l\boldsymbol{I} \right)^{-2}.$$

As $\Phi_l(\boldsymbol{A})$ is convex in l, we have

$$\Phi_{l+\delta_L}(\boldsymbol{A}) - \Phi_l(\boldsymbol{A}) \leq \delta_L \frac{\partial}{\partial l} \Phi_{l+\delta_L}(\boldsymbol{A}) = \delta_L \mathrm{Tr}\left(\boldsymbol{A} - (l + \delta_L)\boldsymbol{I} \right)^{-2}.$$

To bound the other term, we use Claim 18.5.1 to prove

$$\mathrm{Tr}\left((\boldsymbol{A} - (l + \delta_L \boldsymbol{I})^{-1} \right) \leq \frac{1}{1/\Phi_l(\boldsymbol{A}) - \delta_L}.$$

□

So, for there to exist a $\boldsymbol{v}_i$ that we can add to S with scale factor c so that neither barrier function increases, we just need that

$$\frac{1}{\delta_U} + \Phi^u(\boldsymbol{A}) \leq \frac{1}{\delta_L} - \frac{1}{1/\Phi_l(\boldsymbol{A}) - \delta}.$$

If this holds, then there is a $\boldsymbol{v}_i$ so that

$$\boldsymbol{v}_i \boldsymbol{U}_{\boldsymbol{A}} \boldsymbol{v}_i \leq \boldsymbol{v}_i \boldsymbol{L}_{\boldsymbol{A}} \boldsymbol{v}_i.$$

We then set c so that

$$\boldsymbol{v}_i \boldsymbol{U}_{\boldsymbol{A}} \boldsymbol{v}_i \leq \frac{1}{c} \leq \boldsymbol{v}_i \boldsymbol{L}_{\boldsymbol{A}} \boldsymbol{v}_i.$$

We now finish the proof by checking that the numbers I gave earlier satisfy the necessary conditions. At the start both barrier functions are less than 1, and we need to show that this holds throughout the algorithm. At every step, we will have by induction

$$\frac{1}{\delta_U} + \Phi_u(\boldsymbol{A}) \leq \frac{1}{2} + 1 = \frac{3}{2},$$

and

$$\frac{1}{\delta_L} - \frac{1}{1/\Phi_l(\boldsymbol{A}) - \delta_L} \geq 3 - \frac{1}{1 - 1/3} = \frac{3}{2}.$$

So, there is always a $\boldsymbol{v}_i$ that we can add to S and a scaling factor c so that both barrier function remain upper bounded by 1.

If we now do this for $6n$ steps, we will have

$$l = -n + 6n/3 = n \qquad \text{and} \qquad u = n + 2 \cdot 6n = 13n.$$

The bound stated at the beginning of the lecture comes from tightening the analysis. In particular, it is possible to improve Lemma 18.7.2 so that it says

$$\sum_i \boldsymbol{v}_i^T \boldsymbol{L}_{\boldsymbol{A}} \boldsymbol{v}_i \geq \frac{1}{\delta_L} - \frac{1}{1/\Phi_l(\boldsymbol{A})}.$$

I recommend the paper for details.

18.9 Progress and Open Problems

- It is possible to generalize this result to sums of positive semidefinite matrices, instead of outer products of vectors [**?**].
- It is now possible to compute sparsifiers that are almost this good in something close to linear time. [**?**, **?**].
- Given last lecture, it seems natural to conjecture that the scaling factors of edges should be proportional to their weights times effective resistances. Similarly, one might conjecture that if all vectors $\boldsymbol{v}_i$ have the same norm, then the scaling factors are unnecessary. This is true, but not obvious. In fact, it is essentially equivalent to the Kadison-Singer problem [**?**, **?**].